On the Absence of Analytic Continuation of Thermodynamic Potentials at First Order Phase Transitions

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Ce travail est consacré à l'étude des propriétés d'analyticité des potentiels thermodynamiques (énergie libre, pression) pour des systèmes classiques réticulaires à basse température. L'objet central de notre analyse, dans ce cadre, est de montrer rigoureusement l'absence de prolongement analytique aux points de transition de phase du premier ordre.

Notre premier résultat s'applique à la classe générale de modèles à deux phases considérée dans la Théorie de Pirogov-Sinai. L'analyse révèle que la condition de Peierls, hypothèse de base de cette théorie, suffit à montrer l'absence de prolongement analytique de la pression au point de transition.

Dans une deuxième partie, on étudie un cas particulier de potentiel à deux corps, de la forme $\gamma^d J(\gamma x)$, où $\gamma > 0$ est un petit paramètre et J une fonction à support borné (dans la limite $\gamma \searrow 0$, ce potentiel permet de justifier rigoureusement la "loi des aires" de la Théorie de van der Waals-Maxwell). Pour toutes les valeurs strictement positives (petites) du paramètre γ, on montre que l'énergie libre n'a pas de prolongement analytique aux points de transition.

Ces résultats confirment d'anciennes conjectures affirmant que la nature finie de la portée de l'interaction est responsable de la présence de singularités dans les potentiels thermodynamiques.

Foreword

This work is devoted to the study of the analyticity properties of thermodynamic potentials (free energy, pressure) for classical lattice systems at low temperature. The central topic of our analysis, in this framework, is to show rigorously the absence of analytic continuation at points of first order phase transition.

Our first result applies to the general class of two phase models considered in the Theory of Pirogov-Sinai. The analysis reveals that the Peirls condition, which is the basic hypothesis of the theory, suffices to show the absence of analytic continuation of the pressure at the transition point.

In a second part, we study a particular two body potential, of the form $\gamma^d J(\gamma x)$, where $\gamma > 0$ is a small parameter and J a function with bounded support (in the limit $\gamma \searrow 0$, this potential gives a rigorous justification of the "equal area rule" of the van der Waals-Maxwell Theory). For all small strictly positive values of the parameter γ, we show that the free energy has no analytic continuation at the transition points.

These results confirm early conjectures stating that the finiteness of the range of interaction is responsible for the presence of singularities in the thermodynamic potentials.

Remerciements

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Contents

Chapter 1

Introduction

This thesis is devoted to the study of a particular aspect of rigorous statistical mechanics, namely the problem of analytic continuation at first order phase transition. Until the work of S.N. Isakov on the Ising Model, and since the early theories of the condensation phenomenon initiated by van der Waals and Mayer, this had remained an unresolved problem in the field of mathematical physics. Our results are a continuation of those of Isakov, and aim at showing that absence of analytic continuation is generic and valid for any lattice system with finite range interactions.

Since the problem of analytic continuation is related, above all, to the *existence* of transition points, we devote Section 1.1 of this introduction to a brief exposition of the existing theories of the condensation phenomenon. At each step of this brief review, we will emphasise the point of view which will be ours throughout this thesis: the analyticity properties of thermodynamic potentials. These historical references are, necessarily, non-technical and non-exhaustive; our choice has been to mention only a few major developments, which are essentially the Theory of van der Waals (1873), followed, respectively, by the Theories of Mayer (1937), Yang-Lee (1952), and Pirogov-Sinai (1975). Kac potentials are also a method for studying phase transitions and are one of the main subjects of interest of this thesis.

In Section 1.2 we consider finer analyticity properties of thermodynamic potentials at first order phase transition and describe the few existing rigorous results concerning non-analyticity. We expose briefly our results in Section 1.2.4. The rigorous description of lattice models, as well as a more precise formulation of our results, will be given in Chapters 2 and 3, and in Appendix A.

1.1 On the Theory of Condensation

1.1.1 The Thesis of van der Waals

The theory leading to a first description of the phenomenon of condensation started in 1873 with the thesis of J.D. van der Waals [1], who succeeded in establishing an equation of state that described significant deviations from the equation of the perfect gas. His main contribution was to make two fundamental hypotheses on the microscopic structure of matter, that can be formulated as follows:

1. The microscopic constituents of the gas, called molecules, are extended in space; their volume is denoted $b > 0$.

2. The interaction between the molecules consists of two parts: the first is *repulsive*, and forbids any pair of molecules to overlap; the second is *attractive*, *long range*, characterised by a constant $a > 0$ called *specific attraction*.

An important feature of the attractive part of the potential is that it does not depend on the position of the molecules. Assuming the system to be homogeneous, these two hypotheses led van der Waals to his famous equation of state, relating the pressure p, the volume v, and the temperature T of the gas:

$$\Big(p + \frac{a}{v^2}\Big)\Big(v - b\Big) = RT\,, \tag{1.1}$$

where $R = 8,3145\, J/mK$ is the universal gas constant. An isotherm is the family of values taken by the pressure in function of the volume when the temperature is fixed: $p = p(v)$. As a simple analysis shows, the particularity of this equation lies in the fact that there exists a critical temperature $T_c = T_c(a,b)$ such that when $T > T_c$, $p(v)$ is essentially a refinement of the former equation of the perfect gas, but when $T < T_c$, then a new phenomenon occurs. Namely, there exist pressures p_0 such that the cubic equation $p(v) = p_0$ has three distinct roots. At one of these roots, the derivative of the pressure with respect to v is positive which is a contradiction with the principles of thermodynamics. For this value of v, the system is said to be unstable. When $T \nearrow T_c$, the three roots coalesce into a single one, denoted v_c. The point $(v_c, p(v_c))$ is called the critical point.

[1] van der Waals J.D., *De Continuiteit va den Gas en Vloeistoftoestand*, Leiden, (1873). Several translations exist, among which the one of J.S. Rowlinson, *On the Continuity of the Gaseous and Liquid States*, in Studies in Statistical Mechanics, vol. 14, J.L. Lebowitz ed., North-Holland, (1988).

Van der Waals compared his isotherms with those observed experimentally by Andrews [2]. For $T > T_c$, the isotherm (1.1) gave remarkable results. When $T < T_c$, it described correctly the pure gas and liquid phases, but not their coexistence. That is, by starting the system in a gas phase, with large volume, and by progressively decreasing v, one expects to reach a volume v_g, called the *condensation point*, at which the gas is in a state of saturated vapour, with pressure p_{sat}. At $v = v_g$, the vapour starts to condense, and there is creation of a macroscopic fraction of liquid. When v further decreases, the pressure remains constant, $p = p_{sat}$, and the quantity of liquid (resp. gas) phase increases (resp. decreases) linearly with v, until a volume v_l is reached, called the *evaporation point*, at which the vapour is entirely transformed into liquid. For volumes $v \leq v_l$, the system is in a pure liquid phase, and the pressure starts increasing again, much faster, since liquids have a very low compressibility coefficient.

These experimental facts suggest that van der Waals' isotherm $p(v)$ should be replaced, on a well chosen interval $[v_l, v_g]$, by a flat coexistence plateau, removing at the same time the unstable values of v.

At about the same time, Maxwell [3] gave a prescription, today called the Maxwell Construction, for the height at which the plateau p_{sat} should be positioned. By imposing coexistence and assuming liquid and vapour to be at thermal and mechanical equilibrium (i.e. same temperature and pressure), Maxwell was led to define p_{sat} (and, in turn, v_l and v_g), in the following way:

$$p_{sat} \cdot (v_g - v_l) = \int_{v_l}^{v_g} p(v) \mathrm{d}v \,. \tag{1.2}$$

This definition has a geometrical interpretation: p_{sat} must be chosen such that the area of $p(v)$ above and below p_{sat} are equal, therefore bearing the name "equal area rule". (See Figure 1.1 for an illustration of this construction.) The new isotherm is denoted $\mathrm{MC}\, p(v)$. The derivative of $\mathrm{MC}\, p(v)$ has the correct sign for all v, and has jumps at v_l and v_g. The interval $(b, v_l]$ is called the liquid branch; $[v_g, +\infty)$ is called the gas branch.

The Simple Analytic Structure

After the Maxwell Construction, a natural question raises, namely of the significance of $p(v)$ when $v \in [v_l, v_g]$. If we consider the path $v \searrow v_g$, along the gas

[2] Andrews T., *Ann. Physik Ergänzbd.* 5, **64**, (1871). Andrews had made precise measurements on carbonic acid, varying temperature and pressure, and had discovered the existence of a critical point.

[3] Maxwell J.C., *On the Dynamical Evidence of the Molecular Constitution of Bodies*, Nature, (1875). See also The Scientific Papers of James Clerk Maxwell, W.A.Niven ed., 418-438, (1890) (reprinted by Dover, New York, (1965)).

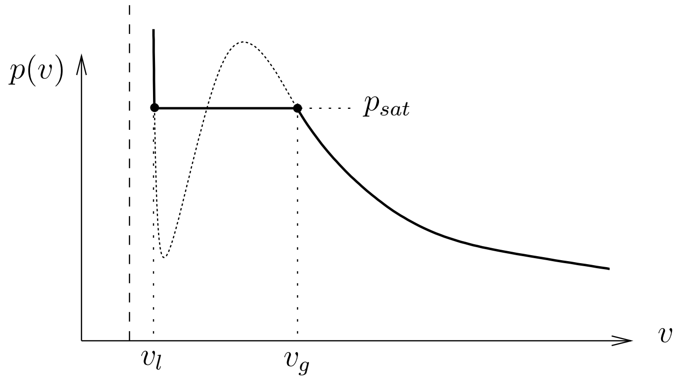

Figure 1.1: The Maxwell Construction.

branch, then MC $p(v)$ can be *continued analytically at* v_g, and the unique analytic continuation is nothing but the original isotherm provided by (1.1). Experimentally, the state obtained by driving the gas along this path without it starting to condensate is called a *metastable state*. Such a state is likely to have a very long lifetime, but the slightest perturbation can abruptly drive the system away from it, into the stable liquid state with same pressure. The values taken by the analytic continuation of MC $p(v)$ on a small interval $(v_g - \epsilon, v_g]$ were originally interpreted as the pressure of what is called a super-saturated vapour. The same can be done along the liquid branch, in which case a liquid can be prepared into a state of under-cooled liquid. Such states can be created in the laboratory; their existence had first been proposed by Thomson [4]. The latter was commenting on Andrews' experiments, and in particular on the isotherms below the critical temperature. Andrews had, essentially, drawn straight lines between the points that represent the coexisting liquid and vapour, on such an isotherm, and Thomson thought that

> [..] *this represented a practical breach of continuity, and there may exist, in the nature of things, a theoretical continuity across this breach having some real or true significance. This theoretical continuity, from the ordinary liquid state to the ordinary gaseous state, must be supposed to be such as to have its various courses passing through conditions of pressure, temperature, and volume in unstable equilibrium for any fluid matter theoretically conceived as homogeneously distributed while passing through the intermediate conditions.*

The original isotherm of van der Waals $p(v)$ can be thought as the "theoretical continuity" mentioned by Thomson.

[4]Thomson J., *Considerations on the Abrupt Change at Boiling or Condensing in Reference to the Continuity of the Fluid State of Matter*, Proc. Roy. Soc. **20**, 1, (1871).

During the later developments of statistical mechanics and for many decades, analytic continuation was believed to be the natural way of defining the thermodynamic potentials of a metastable state. Nowadays, metastability is rather considered as a dynamical problem of non-equilibrium statistical mechanics, but the problem of analytic continuation remains of central importance in the understanding of the structure of thermodynamic potentials. In the van der Waals-Maxwell Theory, the branches of MC $p(v)$ have analytic continuation through the transition points. Our concern is to know if this remarkable property, in general, holds in the context of classical equilibrium statistical mechanics.

The Birth of Equilibrium Statistical Mechanics

The postulates and methods of statistical mechanics were given a definite form by Boltzmann and Gibbs, at the end of the 19th century. In this theory, macroscopic quantities (pressure, magnetisation, etc.) are related precisely to the underlying interactions between the microscopic constituents (molecules), via the notion of statistical ensemble: consider a gas of N indistinguishable particles living in a vessel V. The positions of the particles are denoted $x_1, \dots, x_N$, $x_i \in V$, and the energy of interaction is denoted $U(x_1, \dots, x_N)$. In the canonical ensemble, the number of particles is fixed, and the free energy of the gas, at inverse temperature $\beta > 0$, $f_V = f_V(N, \beta)$, is defined by the identity

$$e^{-\beta V f_V} = \frac{\lambda^N}{N!} \int_V \mathrm{d}x_1 \cdots \int_V \mathrm{d}x_N e^{-\beta U(x_1,\dots,x_N)}\,, \tag{1.3}$$

where the constant λ comes from integration over the momenta of the particles. The quantity

$$Q^{\mathrm{can}}(V, N, \beta) := \frac{1}{N!} \int_V \mathrm{d}x_1 \cdots \int_V \mathrm{d}x_N e^{-\beta U(x_1,\dots,x_N)} \tag{1.4}$$

is called the configurational partition function. In the grand canonical ensemble, the number of particles can fluctuate, and the pressure $p_V = p_V(\mu, \beta)$ depends on the chemical potential $\mu \in \mathbb{R}$. It is defined by the identity

$$e^{\beta V p_V} := \sum_{N \geq 0} e^{\beta \mu N} Q^{\mathrm{can}}(V, N, \beta) \equiv Q^{\mathrm{gcan}}(V, \mu, \beta)\,. \tag{1.5}$$

All the thermodynamic properties of the system are expected to be contained in the functions f_V and p_V. In the following section, we describe the work of the pioneers who tried to understand condensation from these first principles.

1.1.2 The Treatment of Mayer, Kahn et al.

The problem was to know whether the free energy and pressure, as defined in formulas (1.3) or (1.5), can consist of two branches separated by a coexistence plateau, as in Figure 1.1. It was stated explicitly only in the thirties [5] that such sharp behaviour should occur only in the limit of a very large number of particles. This limiting procedure, called the *thermodynamic limit*, was discussed by Kahn [6]:

> *From the mathematical standpoint, it is hard to imagine how from* (1.3) *it can follow that* f_V *(and therefore* p_V*) as a function of* V *consists of three analytically different parts. It seems to us that this is possible because we are only interested in a* limit property *of* f_V*. The problem has a physical sense only when* N *is very large. One may expect that for a fixed specific volume* $v = \frac{V}{N}$ *the limit*
>
> $$f(v, \beta) = \lim f_V(N, \beta) \tag{1.6}$$
>
> *for* $V \to \infty$, $N \to \infty$, $v = \frac{V}{N}$ *fixed, may exist.* [...] *It is not surprising that this function can consist of analytically different parts.*

The limits $f = \lim_{V\to\infty} f_V$ and $p = \lim_{V\to\infty} p_V$ will be called respectively free energy and pressure densities. Their existence and convexity would be shown later by van Hove [7]. As will be seen, the first attempts to describe condensation from first principles were very imprecise in respect to the thermodynamic limit. For this reason, they failed in describing the pure liquid phase, but nevertheless succeeded in giving an argument for the occurence of condensation, which later proved to be correct.

The Treatment of Mayer

The first attempt was made by Mayer [8], in the case where the particles interact with a two body potential:

$$U(x_1, \dots, x_N) = \sum_{1 \le i < j \le n} \phi(x_i, x_j)\,. \tag{1.7}$$

We will assume for simplicity that ϕ depends only on the distance between x_i and x_j, and that ϕ has a finite range, i.e. that $\phi(x_i, x_j) = 0$ when the particles

[5] See for instance Dresden M., *Kramers Contributions to Statistical Mechanics*, Physics Today, september (1988).

[6] Kahn B., *On the Theory of the Equation of State*, thesis, (1938). A re-edition can be found in Studies in Statistical Mechanics, vol.3, J. de Boer G.E. Uhlenbeck eds., North-Holland, (1965).

[7] van Hove L., *Quelques Propriétés Générales de l'Intégrale de Configuration d'un Système de Particules avec Interaction*, Physica XV, **11-12**, 951-961, (1949).

[8] Mayer J.E., *The Statistical Mechanics of Condensing Systems*, Journ. Chem. Phys. **5**, 67-73, (1937).

i and j are sufficiently far apart. Mayer used the same representation that had been obtained by Ursell:

$$Q^{\mathrm{can}}(V,N,\beta) = \sum_{\substack{\{m_1,\dots,m_N\} \\ \sum_l l m_l = N}} \prod_l \frac{(V b_l(V))^{m_l}}{m_l!}\,, \tag{1.8}$$

where the sum is over all possible partitions of N molecules into families of clusters: m_1 clusters of one molecule, m_2 clusters of 2 molecules, etc. The numbers $b_l(V)$ are called *cluster integrals*. The grand canonical pressure then has the following form:

$$\beta p_V(z) = \chi_V(z) := \sum_{l\geq 1} b_l(V) z^l\,, \tag{1.9}$$

where $z = e^{\beta\mu}$ is the fugacity, and the pressure density equals

$$\beta p(z) = \chi(z) := \lim_{V\to\infty} \chi_V(z)\,. \tag{1.10}$$

The cluster integrals can be positive or negative. When V is very large, they become independent of V [9]. Define:

$$b_l^0 := \lim_{V\to\infty} b_l(V)\,. \tag{1.12}$$

Mayer assumed that all the cluster integrals are positive, and neglected their dependence on the volume, replacing each $b_l(V)$, in $Q^{\mathrm{can}}(V,N,\beta)$, by its limiting value b_l^0. Under these hypothesis, he was led to the following expression for the pressure density:

$$\beta p^0(z) = \chi^0(z) := \sum_{l\geq 1} b_l^0 z^l\,. \tag{1.13}$$

[9] Namely, consider the definition of $b_l(V)$, which is

$$b_l(V) = \frac{1}{l!V} \sum_{\substack{\mathcal{G}'\subset\mathcal{G}_l \\ \text{connected}}} \int_V \cdots \int_V \mathrm{d}x_1 \dots \mathrm{d}x_l \prod_{\substack{e\in\mathcal{G}' \\ e=(i,j)}} \left(e^{-\beta\phi(x_i,x_j)} - 1\right), \tag{1.11}$$

where the sum is over all simple non-oriented connected partial graphs $\mathcal{G}'$ of the complete graph with l vertices (denoted $\mathcal{G}_l$). Assume x_1 is fixed. Then, since ϕ has finite range and the graphs $\mathcal{G}'$ must be connected, the integration over $x_2,\dots,x_l$ is essentially independent of x_1, up to a term of order $\frac{\partial V}{V} \simeq V^{-\frac{2}{3}}$. The integration over x_1 then gives V, which cancels with the factor V appearing in the denominator. Clearly, this argument holds when l is small when compared to V. Since l ranges from 1 to N, the approximation holds when $\frac{N}{V}$ is small, i.e. in the gas phase.

The assumption of Mayer is equivalent to exchanging the thermodynamic limit with the sum over $l \geq 1$:

$$\lim_{V\to\infty} \sum_{l\geq 1} b_l(V) z^l = \sum_{l\geq 1} \lim_{V\to\infty} b_l(V) z^l \,. \tag{1.14}$$

(We will see that this operation can be justified when $|z|$ is small enough). To obtain p^0 as a function of the specific volume v, which is determined by $v^{-1} = \rho = \frac{\partial p^0}{\partial \mu} = \beta z \frac{\partial p^0}{\partial z}$, we must find $z = z(v)$, given implicitly by

$$v^{-1} = \sum_{l\geq 1} l b_l^0 z^l \,. \tag{1.15}$$

After that, the pressure of Mayer has the form of an infinite sum:

$$\beta p^0(v) = \frac{1}{v} + \frac{a_1}{v^2} + \frac{a_2}{v^3} + \dots \,. \tag{1.16}$$

The expression obtained by Mayer gave the corrections to the equation of state of the perfect gas (for which $a_i = 0$ for all $i \geq 1$), and showed that, in general, an equation of state has no reason for being given in a closed form, like van der Waals had obtained. Since Mayer's equations should be considered as valid in the pure gas phase, it remains to see if condensation can be understood from the infinite series (1.13) and (1.16).

Condensation

Let us assume that the series $\sum b_l^0 z^l$ has a finite radius of convergence, denoted as R. Then $\chi^0(z)$ has at least one singularity on the boundary of the disc $|z| \leq R$. Since Mayer assumed that all the coefficients b_l^0 were positive, the point $z^* := R$ is a singularity [10]. Mayer interpreted z^* as the manifestation of condensation, and the value v^* for which $z(v^*) = z^*$ as the condensation point. He also sketched an argument to justify the appearance of the coexistence plateau: for all values of the volume $v \in [0, v^*]$, the pressure remains constant (see Figure 1.2).

A few years later, Kahn and Uhlenbeck [11] were led to the same expression for the pressure, but with a rigorous argument. Rather than assuming positivity of the coefficients b_l^0, they gave a general result, valid under a set of natural assumptions on the function $\chi^0(z)$, and obtained the same isotherm as depicted in Figure 1.2. Although they firmly believed that both phases can be described within the formalism of Gibbs, they also were not successful in describing the pure liquid

[10] See Remmert R., *The Theory of Complex Functions*, Springer, (1991).

[11] Kahn B., Uhlenbeck G.E., *On the Theory of Condensation*, Physica **5**, 399-416, (1938).

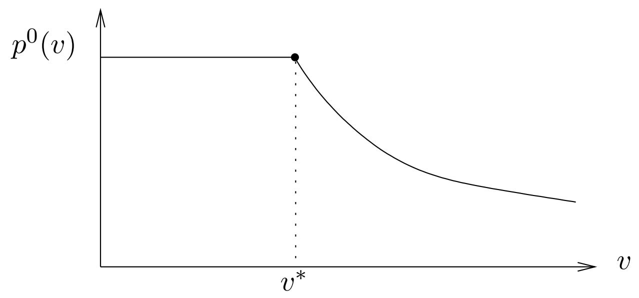

Figure 1.2: A sketch of the isotherm obtained by Mayer et al. Notice the absence of the evaporation point from the theory.

phase within the same equation. Similar computations led Born and Fuchs [12] to the same results.

Naturally, the problems encountered by these authors in describing the liquid phase can be ascribed to the simplifying hypothesis they made on the cluster integrals b_l; it seems very unlikely for the liquid phase to suddenly appear from an equation of state which is valid under hypothesis that hold only in the gas phase. Moreover, the confusion concerning the thermodynamic limit is evident: the function $Q^{\text{can}}(V, N, \beta)$ describes a finite system, but the cluster integrals are approximated by their infinite-volume limit b_l^0. These remarks also raise the question, more serious, of knowing if the singularity z^* proposed by Mayer et al. even determines the *real* condensation point. We will come back to this later.

1.1.3 The Theory of Yang and Lee

After the exact computation of Onsager [13], the major contribution to the theory of Phase Transitions was the double paper of Yang and Lee [14]. (The Peierls argument will be discussed in the following section.) Consider the grand canonical partition function $Q^{\text{gcan}}(V, z, \beta)$ expressed as a function of the fugacity $z \in \mathbb{C}$. Yang and Lee studied the relationship between the zeroes of $Q^{\text{gcan}}(V, z, \beta)$ and the analyticity properties of the pressure density $p(z) = \lim_{V\to\infty} p_V(z)$. The main theorem of their first paper is a general result on the *absence* of phase transitions

[12] Born M., Fuchs K., *The Statistical Mechanics of Condensating Systems*, Proc. Royal Soc. **166**, 391-414, (1938).

[13] Onsager L., *Crystal Statistics I, A Two-Dimensional Model with an Order-Disorder Transition*, Phys. Rev. **65**, 117-149, (1944).

[14] Yang C.N., Lee T.D., *Statistical Theory of Equations of State and Phase Transitions. I. Theory of Condensation*, Phys. Rev. **87**, 3, 404-409, (1952). Lee T.D., Yang C.N., *Statistical Theory of Equations of State and Phase Transitions. II. Lattice Gas and Ising Model*, Phys. Rev. **87**, 3, 410-419, (1952).

in the thermodynamic limit:

Assume there exists a domain $D \subset \mathbb{C}$ *such that* $Q^{\rm gcan}(V, z, \beta) \neq 0$ *for all* $z \in D$, *for all* V. *Then* $p(z)$ *exists and* $z \mapsto p(z)$ *is analytic in* D. *Moreover, the thermodynamic limit and the derivatives with respect to* $\log z$ *commute in* D*:*

$$\lim_{V\to\infty} \frac{\partial^k}{\partial(\log z)^k} p_V(z) = \frac{\partial^k}{\partial(\log z)^k} p(z)\,. \tag{1.17}$$

This result allows to relate the occurence of condensation to the distribution of zeros of the partition function. By definition, the roots of $Q^{\rm gcan}(V, z, \beta)$ always lie outside the positive real line. The previous theorem implies that if the pressure p has, for example, a first order phase transition at a point $z_0 > 0$, then zeros of the grand canonical partition function *must* accumulate at z_0 when V becomes large. This clarifies the role played by the thermodynamic limit, and allows, a priori, a description of the gas *and* liquid phases in the thermodynamic limit.

What the theory does not say is: 1) *where* the zeros accumulate (if they do), and 2) whether the accumulation of zeros at some $z_0 > 0$ *does* induce a phase transition at z_0. Indeed, it could happen that zeros accumulate while the pressure remains analytic. A deep result partially answers the first question. It is the content of the *Unit Circle Theorem* of the second paper:

For the Ising Model [15], $z = e^{\beta h}$, $D = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z| > 1\}$.

That is, the Ising Model can have a phase transition only at $h = 0$. In their second paper, Yang and Lee also established clearly the relationship existing between the Ising Model and the discrete version of the gas, called lattice gas. The Circle Theorem thus allows a precise localisation of the phase transition point of the lattice gas. We will revert to the correspondence ferromagnet-lattice gas in Chapter 2.

1.1.4 Peierls Argument and Pirogov-Sinai Theory

The first proof of the existence of a phase transition originated with the argument given by Peierls in 1936 [16]. In his study of the two dimensional Ising Model, Peierls used a key ingredient that was absent in the works of Mayer, Kahn et al.:

[15] Ising E., *Beitrag zur Theorie der Magnetismus*, Zeits. für Physik **31**, 253, (1925).

[16] Peierls R., *On Ising's Model of Ferromagnetism*, Math. Proc. Cambridge Phil. Soc. **32**, 477-481, (1936).

symmetry. The crucial remark made by Peierls is that in the Ising Model, regions of opposite spin are separated by energy barriers, called *contours*. His idea was then to describe the system in terms of contours rather than configurations, and to show that at low temperature, the volume enclosed by the contours can't exceed a quarter of the total volume of the system. Since the system is invariant under spin-flip, the volume enclosed can contain either + spins or − spins. That is, two phases can be described under the same thermodynamic conditions: the + (resp. −) phase consists of a "sea" of + (resp. −) spins with sparse "islands" of − (resp. +) spins. From the point of view of the lattice gas, this argument provides the first rigorous construction of stable isotherms showing condensation and evaporation [17].

A few decades later, Pirogov and Sinai [18] took advantage of the contour picture proposed by Peierls and elaborated the first general theory for the study of first order phase transitions, applicable to a broad class of lattice spin systems, with multiple coexisting phases. The starting point of their theory is the consideration of a hamiltonian H_0 having q degenerated ground states $\psi_1, \dots, \psi_q$. These ground states need not be related by symmetry. The main hypothesis on H_0 is that it satisfies the *Peierls condition*; that is, regions of distinct ground states are separated by energy barriers, the contours. The aim of the theory is to describe the phase diagram of the perturbed hamiltonian

$$H^{\underline{\mu}} = H_0 + \sum_{i=1}^{q-1} \mu_i H_i \,, \tag{1.18}$$

where $\underline{\mu} = (\mu_1, \dots, \mu_{q-1}) \in \mathbb{R}^{q-1}$ and the family $H_1, \dots, H_{q-1}$ splits the degeneracy of H_0. In the simple case $q = 2$, where two phases coexist, the main result is the following:

Let $\mathfrak{p} = \mathfrak{p}(\mu)$, $\mu \in \mathbb{R}$, denote the pressure [19] of the model with hamiltonian H^μ. At low temperature, there exists a transition point $\mu^* = \mu^*(\beta) = O(e^{-\beta})$ such that $\mathfrak{p}$ has a phase transition at μ^*. For $\mu \geq \mu^*$, the phase 1 is stable, and for $\mu \leq \mu^*$ the phase 2 is stable. At μ^* the two phases coexist. The phase q, $q = 1, 2$, describes small stable deviations (the "islands") from the ground state ψ_q (the "sea").

[17] It is surprising to notice that the work of Peierls, which appeared *before* the theories of Mayer, Kahn et al., was ignored, until refinements of the argument were given by Griffiths and Dobrushin in the sixties. Even Yang and Lee did not mention it in their double paper of 1952.

[18] Pirogov S.A., Sinai Y.G., *Phase Diagrams of Classical Lattice Systems*, Teoreticheskaya i Matematicheskaya Fizika **26**, 1, 61-76, (1976). See also the book of Sinai, *Theory of Phase Transitions: Rigorous Results*, Pergamon Press, (1982).

[19] In the literature, this function is usually called "free energy". We keep the name "free energy" for the thermodynamic potential that depends on the density. The letters $\mathfrak{f}$, $\mathfrak{p}$ will be used for spin systems, and f, p will be used for gases and liquids.

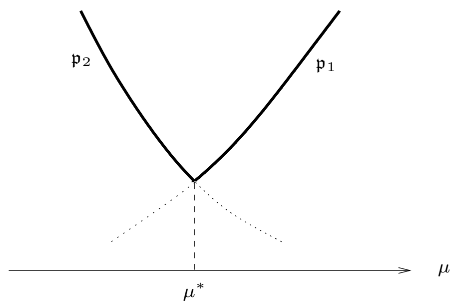

Figure 1.3: The construction of the pressure for a two phase system, in the alternate version of Zahradník. The point where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ coincide is the transition point μ^*. $\mu > \mu^*$ corresponds to the pure phase 1, $\mu < \mu^*$ to the pure phase 2.

Two important features are: 1) the theory depends weakly on the details of the interactions (anisotropies, multibody interactions, etc.), and 2) the different phases are treated in an equivalent manner, even in the absence of symmetry. These two points show that, conceptually, the Pirogov-Sinai Theory is an essential step forward in the understanding of phase transitions, and represents much more than just a "generalisation" of the Peierls argument, as can often be read.

A decade later, Zahradník [20] proposed an alternate way of constructing the phase diagram, using the notion of *truncated models*. These essentially consist of associating each ground state ψ_q, $q = 1, 2$, with a truncated pressure $\mathfrak{p}_q$ obtained by considering a model in which only stable deviations from the ground state ψ_q are allowed. Then, the real pressure of the model is obtained by finding the maximal truncated pressure (see Figure 1.3):

$$\mathfrak{p} = \max\{\mathfrak{p}_1, \mathfrak{p}_2\}\,. \tag{1.19}$$

The construction of Zahradník was simplified and extended to complex interactions by Borgs and Imbrie [21]. In the complex case, the refinement of the previous theorem is the following:

The pressure $\mathfrak{p} = \mathfrak{p}(\mu)$ *is analytic on* $\{\mu < \mu^*\}$ *and* $\{\mu > \mu^*\}$.

An interesting contribution was the work of Borgs and Kotecký [22], who showed that the truncated pressures could be made C^k in μ when β is large enough

[20] Zahradník M., *An Alternate Version of Pirogov-Sinai Theory*, Commun. Math. Phys. **93**, 559-581, (1984).

[21] Borgs C., Imbrie J.Z., *A Unified Approach to Phase Diagrams in Field Theory and Statistical Mechanics*, Commun. Math. Phys. **123**, 305-328, (1989).

[22] Borgs C., Kotecký R., *A Rigorous Theory of Finite-Size Scaling at First-Order Phase Transitions*, J. Stat. Phys. **61**, 79-119, (1990).

(depending on k). By (1.19), the functions $\mathfrak{p}_1, \mathfrak{p}_2$ thus provide C^k-continuations of the pressure across μ^* (see figure 1.3). Although these continuations can be made as smooth as desired by taking β large enough, one of our main results (see Theorem 1.2 of Section 1.2.4) will show that *analytic* continuation is never possible.

Recently, Biskup et al. [23] showed that the smooth truncated pressures of Borgs and Kotecký could be used to generalise the Circle Theorem of Yang and Lee to the class of models treated by the Pirogov-Sinai Theory. Finally, the truncation was used also by Lebowitz, Mazel, Presutti [24] in their recent proof of a phase transition in the continuum.

1.1.5 Kac Potentials

After the contributions of Mayer et al., Peierls and Yang-Lee, it remained to be understood how exactly the van der Waals-Maxwell Theory, in particular the Maxwell Construction, could be justified from first principles.

On one hand, we must keep in mind the second assumption of van der Waals, i.e. that the particles attracted each other with a weak infinite range potential, independent of the distance between the molecules, proportional to the square of the density $\rho^2 = v^{-2}$. This approximation produces the loop in the low temperature isotherms, and is the reason for which the Maxwell Construction was necessary. On the other hand, van Hove [25] showed that when the range of interaction is finite (possibly very long) the free energy as defined in (1.6) is always convex. That is, the pressure is always a decreasing function of the specific volume and need not be complemented by a Maxwell Construction.

To interpolate between finite range and van der Waals-type interactions, Kac, Uhlenbeck and Hemmer [26] studied a one-dimensional model in which the range of the interaction is a parameter of the model. In the limit where the range of interaction goes to infinity, their theory gave the first rigorous justification of the Maxwell Construction. Their result was generalised to higher dimensions by Lebowitz and Penrose [27], and can be described as follows. Let $\varphi : \mathbb{R}^d \to \mathbb{R}^+$ be

[23] Biskup M., Borgs C., Chayes J.T., Kleinwaks L.J., Kotecký R., *Partition Function Zeroes at First Order Phase Transition: A General Analysis*, preprint, (2003).

[24] Lebowitz J.L., Mazel A., Presutti E., *Liquid-Vapor Phase Transitions for Systems with Finite-Range Interactions*, J. Stat. Phys. **94**, 5-6, 955-1025, (1999).

[25] van Hove L., *Quelques Propriétés Générales de l'Intégrale de Configuration d'un Système de Particules avec Interaction*, Physica XV, **11-12**, 951-961, (1949).

[26] Kac M., Uhlenbeck G.E., Hemmer P.C., *On the van der Waals Theory of the Vapor-Liquid Equilibrium*, J. Math. Phys. **4**, 2, 216-228, (1962).

[27] Lebowitz J.L., Penrose O., *Rigorous Treatment of the van der Waals-Maxwell Theory of*

supported by the cube $[-1,+1]^d$ (d is the dimension of the system), so that

$$\int \varphi(x)\mathrm{d}x = \alpha\,. \tag{1.20}$$

Let $0 < \gamma < 1$ be the *scaling parameter*, and define the *Kac potential*

$$K_\gamma(x,y) := \gamma^d \varphi(\gamma(x-y))\,. \tag{1.21}$$

The potential considered by the the authors is of the form

$$\phi_\gamma(x,y) = q(x,y) - K_\gamma(x,y)\,, \tag{1.22}$$

where $q(x,y)$ is a short range repulsive potential equal to $+\infty$ if $x-y$ is smaller than the diameter of the particles, and zero if otherwise. When γ is small, K_γ is weak, with range $R = \gamma^{-1}$. One thus expects systems with Kac potentials to have properties analogue to those predicted by the van der Waals Theory.

Lebowitz and Penrose considered the double limiting process which consists first in taking the thermodynamic limit $V \to +\infty$, and *then* the limit $\gamma \searrow 0$. This limiting procedure corresponds to $1 \ll \gamma^{-1} \ll V$. Since, in this procedure, the limit $\gamma \searrow 0$ is taken *after* the thermodynamic limit, the range of interaction of the Kac potential can always be considered as very small when compared to the size of the vessel V. The limit $\gamma \searrow 0$ is called the *van der Waals Limit*.

Let $f_\gamma = f_\gamma(\rho)$ denote the free energy density of the model, obtained after the thermodynamic limit $V \to \infty$. According to the Theorem of van Hove, f_γ is convex in ρ for all $\gamma > 0$. Then, the van der Waals Limit gives a function $\lim_{\gamma\searrow 0} f_\gamma$ which is still convex. The main result is the following.

Let $\tilde{f}$ denote the free energy of the reference system, i.e. with potential $q(x,y)$ rather than $\phi_\gamma(x,y)$. In the van der Waals Limit, $f_\gamma(\rho)$ converges to the convex envelope of $-\frac{1}{2}\alpha\rho^2 + \tilde{f}(\rho)$ (see Figure 1.4).

Notice that taking the convex envelope is equivalent, in terms of the pressure, to applying the Maxwell Construction.

The method used by Lebowitz and Penrose to show this result (refered to as the Lebowitz-Penrose Theorem in future) was inspired by a coarse-graining procedure invented by van Kampen [28]: the volume occupied by the system is divided into a large number of cells, each small compared with the range of the long range

the Liquid-Vapor Transition, J. Math. Phys. **7**, 1, 98-113, (1966).

[28] van Kampen N.G., *Condensation of a Classical Gas with Long-Range Attraction*, Phys. Rev. **135**, A362-369, (1964).

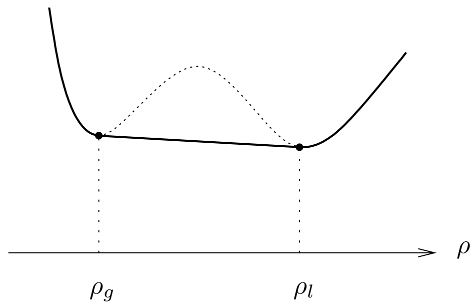

Figure 1.4: The Lebowitz-Penrose Theorem: in the van der Waals Limit, the free energy f_γ converges to the convex envelope of $-\frac{1}{2}\alpha\rho^2 + \tilde{f}(\rho)$. The points ρ_g, ρ_l are the points of condensation and evaporation.

attractive force, but large enough to contain many particles. The distribution of the molecules throughout the cells is then determined by minimising the free energy. In a pure phase (liquid or gas branch), the density is uniform, but in the coexistence region the free energy is found to be minimised by non-homogeneous distribution throughout the cells. This is the mechanism responsible for the appearance of the plateau in the free energy.

From the point of view of analyticity, this theory makes the same predictions as the one of van der Waals-Maxwell. After the van der Waals Limit, thermodynamic potentials are analytic in a pure phase, and can be continued analytically at condensation/evaporation points.

We will come back to a detailed description of Kac potentials in Chapter 3.

1.2 Analytic Properties at Condensation

All the theories we described in the previous section agree that in a pure phase, thermodynamic potentials are analytic. As to analyticity *at* transition points, only the van der Waals-Maxwell Theory and Kac potentials in the van der Waals Limit give precise information. As we saw, the problem of analytic continuation at first order phase transition points was originally related to the existence of metastable states. Another motivation for the study of analytic properties is the following discussion. It originates with the problem of knowing if the theories of Mayer and Yang-Lee describe the same condensation point. Consider the expansion obtained by Mayer for the pressure:

$$\chi^0(z) = \sum_{l\geq 1} b_l^0 z^l \,. \tag{1.23}$$

Mayer actually discussed condensation without considering the problem of convergence. The first proof of the convergence of the series (1.23) for sufficiently small z was given by Groeneveld [29]. We present here a stronger result that can be found in the book of Ruelle [30]:

Assume $B := \inf_x \phi(0,x) > -\infty$, *and*

$$C(\beta) := \int |e^{-\beta\phi(0,x)} - 1| \mathrm{d}x < \infty \,. \tag{1.24}$$

Then the radius of convergence of the series $\sum b_l^0 z^l$ *is* at least $e^{-2\beta B - 1} C(\beta)^{-1}$. *In the disc* $\{|z| < e^{-2\beta B-1}C(\beta)^{-1}\}$, *the pressure of the gas is given by the Mayer expansion:* $\beta p(z) = \chi(z) \equiv \chi^0(z)$.

That is, well inside the gas phase (small z), the pressure has a convergent Taylor expansion, and the interchange (1.14) made by Mayer is justified. It is to be remembered that Mayer assumed all the cluster integrals were positive. In fact, Groeneveld showed that the coefficients b_l^0 have alternating signs: $(-1)^{l+1} b_l^0 \geq 0$. This implies that $\chi^0(z)$, if it has a radius of convergence R, is guaranteed to have a singularity only at the point $z = -R$, which is non-physical. The singularity must be determined by other means. Let us consider the analytic continuation of $\chi^0(z)$ along the positive real line, possibly beyond $z = R$, and denote this analytic continuation by $\tilde{\chi}(z)$. Let us then assume that $\tilde{\chi}(z)$ has a singularity on the positive real line, denoted $z_M > 0$. What must be understood is whether the physical phenomenon of condensation really occurs at z_M. Indeed, the function $\tilde{\chi}(z)$ might very well not "see" the real condensation point, situated somewhere between the origin and the singularity z_M.

On the other hand, we saw that the main result of Yang and Lee indicates that condensation implies accumulation of zeros of the grand canonical partition function. Let z_{LY} denote the smallest point of the positive real line at which zeros accumulate. We thus have two possible a priori condensation points, z_M and z_{LY}. Consequently, the following two questions arise:
Do the points z_M *and* z_{LY} *coincide? Which of them describes condensation?*
Since $\sum b_l^0 z^l$ concides with the real pressure near $z = 0$, we necessarily have $z_{LY} \leq z_M$. It remains to be seen whether $z_{LY} < z_M$ is possible. If so, and if condensation occurs at z_{LY}, then $\tilde{\chi}(z)$, obtained from the theory of Mayer, provides the analytic continuation of the pressure across the condensation point. The possibility of this scenario was suggested in the fifties by Katsura and Fu-

[29] Groeneveld J., *Two Theorems on Classical Many-Particle Systems*, Phys. Letters **3**, 1, 50-51, (1962).

[30] Ruelle D., *Statistical Mechanics. Rigorous Results*, World Scientific, (1969).

jita [31], showing that the problem of analytic continuation at condensation is also intimately related to the validation of the theory of Mayer. Indeed, if it can be shown rigorously, for a simple model, that there is *no* analytic continuation at the condensation point z^*, along $z \nearrow z^*$, then necessarily $z^* = z_{LY} = z_M$, which validates Mayer's determination of the condensation point. In case there *is* analytic continuation at z^*, then Mayer's theory describes only metastable continuation up to some supersaturation limit.

This discussion shows that a detailed investigation of the analyticity properties at condensation should be undertaken, with emphasis on the role played by the physical phenomenon itself. Beyond the possibility of proving absence or presence of metastable states or justifying the Theory of Mayer, the study of analytic properties is a way of understanding the *global structure* of thermodynamic potentials; the question arises as to whether the different branches can be continued analytically, or if the singularities at transition points are such that these continuations are forbidden. In the following two sections, we expose the first two arguments that were given in view of answering this question.

1.2.1 Analytic Continuation in the Mean Field Model

In the mid-fifties, Temperley [32] and Katsura [33] considered the so-called Bragg-Williams approximation of the lattice gas, in which exact computations can be carried out. Today, this model is called the Curie-Weiss or mean field model: all particles interact with each other, although very weakly, as in van der Waals Theory. Temperley and Katsura observed a kink of the pressure at the value $z^* = z_{LY}$, but the Mayer expansion showed a singularity at a point z_M strictly larger than z^* (see Figure 1.5). Within this example, the series expansion of Mayer thus fails in describing the correct condensation point, and provides the analytic continuation of the isotherm across the condensation point - exactly like in van der Waals Theory. It was conjectured by Katsura [34] that *these properties hold in general* for simple models with finite range interactions, like the Ising Model.

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[31] Katsura S., Fujita H., *Point of Condensation and the Volume Dependency of the Cluster Integrals*, Progr. Theor. Phys. **4**, vol. 4, (1951). See also Ikeda K., *On the Theory of Condensation*, Progr. Theor. Phys. **4**, vol. 16, (1956) for a detailed discussion of this problem.

[32] Temperley H.N.V., *The Mayer Theory of Condensation Tested Against a Simple Model of the Imperfect Gas.*, Proc. Phys. Soc. **A 67**, 233-238, (1954).

[33] Katsura S., *Phase Transition of Husimi-Temperley Model of Imperfect Gas*, Progr. Theor. Phys. **6**, vol. 13, (1955).

[34] Katsura S., *On the Theory of Condensation*, Journ. Chem. Phys. **22**, 1277, (1954). The same conjecture appeared again almost ten years later in Katsura S., *Singularities in First-Order Phase Transitions*, Adv. Phys. **12**, 48, 391-420, (1963).

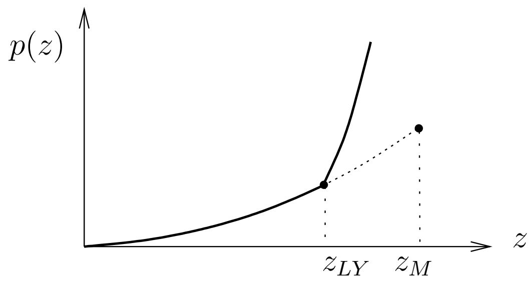

Figure 1.5: The isotherm obtained with the Bragg-Williams approximation of the lattice gas, showing that the transition point z_M of Mayer lies beyond the condensation point $z^* = z_{LY}$.

1.2.2 Non-Analyticity and the Droplet Mechanism

In the sixties, Andreev [35], Fisher [36] and Langer [37] presented an argument saying that for finite range interaction, there is *absence* of analytic continuation at the transition point. They proposed an effective model which captured the main features of the phenomenon of condensation, and characterised in a more precise manner the nature of the singularity present in the pressure. We present this argument in its simplest form, starting with the following heuristic description, given by Fisher:

> *One may get a physical idea as to the "cause" of condensation, however, by considering a real gas or, for that matter, a lattice gas or Ising Model, at low densities and temperature. Evidently, most configurations of the system will consist of distributions of isolated molecules well separated from one another. There will also be present, however, clusters of two, three or more molecules bound together more-or-less tightly by the attractive forces but isolated, for the most part, from other clusters. Clusters of different sizes will be in mutual statistical equilibrium, associating and dissociating, but even fairly large clusters resembling "droplets" of the liquid phase will have some, generally rather small, chance of occurring.*

The simplest definition of the droplet model is the following. Consider the d-dimensional simple lattice gas. Let C be a cluster or "droplet" of $|C| = n$ molecules, with a boundary of length $|\partial C|$ (see Figure 1.6). Let $\tau_0 > 0$ denote the "surface tension" associated to the boundary of the droplet: the surface energy of C is equal to $\tau_0|\partial C|$. Since droplets must tend to minimise their surface, one

[35] Andreev A.F., *Singularity of Thermodynamic Quantities at a First Order Phase Transition*, Soviets Physics JETP, **5**, vol. 18, 1415-1416, (1964).

[36] Fisher M.E., *The Theory of Condensation and the Critical Point*, Physics, **5**, vol. 3, 255-283, (1967).

[37] Langer J.S., *Theory of the Condensation Point*, Annals of Physics, **41**, 108-157, (1967).

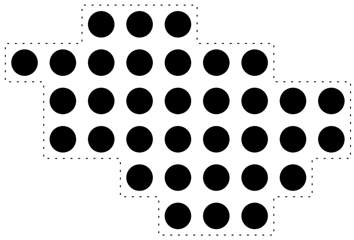

Figure 1.6: A two-dimensional cluster of molecules, or "droplet", with $|C| = 34$ molecules and a boundary $|\partial C| = 30$.

can guess that they have a cubic shape, i.e. $|\partial C| \simeq 2d|C|^{\frac{d-1}{d}}$. We thus define the energy of a droplet by $E(C) := -\mu|C| + \tau_0 2d|C|^{\frac{d-1}{d}}$, where μ is the chemical potential. The Boltzmann factor of a droplet becomes

$$e^{-\beta E(C)} = e^{-\tau|C|^\sigma} z^{|C|} , \tag{1.25}$$

where $\sigma = \frac{d-1}{d}$, $\tau = 2d\beta\tau_0$, and $z = e^{\beta\mu}$ is the fugacity. The transition point is expected to be $\mu^* = 0$, i.e. $z^* = 1$. We then ignore the interactions between the droplets, and consider the pressure defined by (compare with (1.13))

$$p_D(z) := \sum_{n\geq 1} e^{-\tau n^\sigma} z^n . \tag{1.26}$$

Now the integer n is identified with a cubical droplet of volume n. As Langer said,

> [...], *the droplet model is more nearly a phenomenology than a precise model of a physical system. From one point of view, it is simply a guess concerning the asymptotic behaviour of the Mayer cluster coefficients for very large clusters. From another point of view the model is a crude but intuitively appealing microscopic picture of how a macroscopic system might look near its condensation point and well below its critical temperature. In any case, the droplets themselves are not unambiguously definable physical entities whose properties (shape, size, surface free energy, etc.) may be computed systematically from a partition function.*

The radius of convergence of the series (1.26) is $R = 1$. Since the coefficients $e^{-\tau n^\sigma}$ are positive, the point $z^* = 1$ must be a singularity of the function $p_D(z)$. To understand the precise nature of this singularity we make a local analysis of the derivatives of the pressure at $z = z^* = 1$. It is easy to see that the limits $p_D^{(k)}(1^-) := \lim_{z\nearrow 1^-} p_D^{(k)}(z)$ exist for all integer k, and equal

$$p_D^{(k)}(1^-) = \sum_{n\geq k} n(n-1)(n-2)\dots(n-k+1)e^{-\tau n^\sigma} . \tag{1.27}$$

That is, p_D has left derivatives of all orders at $z = 1$. It is often said that the singularity at $z = 1$ is "too weak to be observable", since none of the derivatives diverges when approaching the transition point.

Since all the terms of (1.27) are positive, one can obtain an explicit lower bound on $p_D^{(k)}(1^-)$ by keeping only the droplet corresponding to the term $n = n_0(k)$, defined by

$$n_0(k) := \left\lfloor \left(\frac{k}{\tau\sigma}\right)^{\frac{1}{\sigma}} \right\rfloor . \tag{1.28}$$

An upper bound can be obtained easily; this gives the existence of two constants $C_\pm = C_\pm(\tau, \sigma) > 0$ such that

$$C_-{}^k k!^{\frac{d}{d-1}} \leq p_D^{(k)}(1^-) \leq C_+{}^k k!^{\frac{d}{d-1}} . \tag{1.29}$$

As a consequence, the Taylor series describing p_D in a neighbourhood of $z = 1$, given by

$$p_D(1) + p_D^{(1)}(1^-)(z-1) + \frac{1}{2!} p_D^{(2)}(1^-)(z-1)^2 + \dots \tag{1.30}$$

diverges for all $z \neq 1$, even if $|z - 1|$ is very small.

More than a decade later, Kunz and Souillard [38] analysed a similar model in the context of Bernoulli percolation, where the cluster containing the origin plays the role of the droplet:

$$p_{KS}(z) := \sum_{n \geq 1} \mathbb{P}_p(|C| = n) z^n ; \tag{1.31}$$

$\mathbb{P}_p$ denotes the Bernoulli measure with parameter $p \in [0, 1]$. The difference between this model and the previous one is that the coefficients of the series are not given explicitly; Kunz and Souillard showed that their behaviour is the following (α and τ both depend on p):

$$\mathbb{P}_p(|C| = n) \sim e^{-\alpha n} \qquad \text{for } p \text{ close to } 0 , \tag{1.32}$$

$$\mathbb{P}_p(|C| = n) \sim e^{-\tau n^{\frac{d-1}{d}}} \qquad \text{for } p \text{ close to } 1 , \tag{1.33}$$

For small p, the radius of convergence of (1.31) is thus larger than one, and p_D behaves analytically at $z = 1$ whereas for p close to 1, the asymptotic behaviour of the coefficients is the same as those given a priori in the series (1.26). The

[38] Kunz K., Souillard B., *Essential Singularity and Asymptotic Behavior of Cluster Size Distribution in Percolation Problems*, Journ. Stat. Phys. **19**, 77-106, (1978).

authors were thus led to the same conclusion as before: the pressure has left derivatives of all orders at $z = 1$ but has no analytic continuation.

These simple models suggest that it is the condensation phenomenon itself that is responsible for the absence of analytic continuation: when $|z| < 1$, i.e. in a pure phase, droplets of the wrong phase are strongly suppressed due to the volume term z^n. At $z = 1$, the volume term equals 1, and the decrease of the weight of each droplet is ruled only by the surface term, of the order $e^{-\tau n^\sigma}$. Upon derivation at $z = 1$, the volume term produces essentially a factor n^k. Then, since *all terms of the series are positive*, we can *choose* a single droplet to obtain a lower bound on $p_D^{(k)}(1^-)$; this droplet is chosen so as to maximise the factor $n^k e^{-\tau n^\sigma}$. Hence the choice of the dominant term $n = n_0(k)$, leading to the behaviour $\sim k!^{\frac{d}{d-1}}$ for the derivatives at $z = 1$.

This mechanism seemed to be a reasonable heuristic description of any two phase model with finite range interactions. Griffiths had said, in an earlier discussion on the droplet model [39]:

> [...] *This has led to the suggestion, rather hotly debated among the small number of people who worry about such things, that a similar singularity exists in the actual Ising Model.*

There were thus, apparently, two possible scenarios concerning the analytic behaviour of the pressure of the Ising Model $\mathfrak{p} = \mathfrak{p}(h)$ at $h = 0$ (or, equivalently, of the simple lattice gas at z^*), the first being predicted by mean field, the second by droplet approximations. The solution to this problem would not be given until twenty years later, by Isakov. In the mean time, the contributions to this field were essentially refinements of the droplet model, studies of simplified models, or numerical simulations.

Newman and Schulman [40] made some conjectures for the Ising Model. Their first two conjectures, which to date remain unproved, assert that $\mathfrak{p}$ can be continued analytically *around* $h = 0$, and their third conjecture asserted that there is analytic continuation *through* $h = 0$. Then, Enting and Baxter [41] investigated numerically the series for the magnetisation:

$$m(0) + m'(0)h + \frac{1}{2!}m''(0)h^2 + \dots \,, \tag{1.34}$$

[39] Griffiths R.B., *Phase Transitions*, in Statistical Mechanics and Quantum Field Theory, Les Houches 1970, De Witt C. & Stora R. eds., Gordon & Breech Science Publishers, (1970).

[40] Newman C.M., Schulman L.S., *Metastability and the Analytic Continuation of Eigenvalues*, J. Math. Phys. **18**, 1, 23-30, (1977).

[41] Enting I.G., Baxter R.J., *An Investigation of the High-Field Series Expansions for the Square Lattice Ising Model*, J. Phys. A: Math. Gen. **13**, 3723-3734, (1980).

for different subcritical temperatures, up to 35th order, and observed evidence for divergence of the series, in accordance with the droplet predictions. Similar results were obtained by Privman and Schulman [42], but no final conclusion could be drawn.

1.2.3 The Theorem of Isakov

In 1984, Isakov [43] showed that the singularity predicted by the droplet model indeed occurs, and invalidated the mean field predictions, by showing rigorously that the pressure of the Ising Model has no analytic continuation at $h = 0$.

Theorem 1.1 (Isakov, 1984). *Let $d \geq 2$, β be sufficiently large. Then for all $k \in \mathbb{N}$, the limit of the k-th derivative of the pressure of the Ising model along $h \searrow 0^+$ exists:*

$$\mathfrak{p}^{(k)}(0^+) = \lim_{h \searrow 0^+} \mathfrak{p}^{(k)}(h)\,. \tag{1.35}$$

Moreover, there exist two strictly positive functions $C_\pm = C_\pm(\beta)$, such that for large enough k,

$$C_-{}^k k!^{\frac{d}{d-1}} \leq |\mathfrak{p}^{(k)}(0^+)| \leq C_+{}^k k!^{\frac{d}{d-1}} \tag{1.36}$$

For the lattice gas, this result definitely ruled out the possibility of defining metastability by analytic continuation through $h = 0$. With regard to what was said at the beginning of the section, it also justified the determination of the condensation point made by Mayer. By confirming droplet predictions, the theorem of Isakov showed that the analytic behaviour of thermodynamic potentials at transition points must be intimately related to the range of interactions: very long range interactions (mean field) imply analytic continuation, short range interaction, apparently, imply absence of analytic continuation.

Consider (1.36). The lower bound shows that the series

$$\mathfrak{p}(0) + \mathfrak{p}^{(1)}(0^+)h + \frac{1}{2!}\mathfrak{p}^{(2)}(0^+)h^2 + \dots \tag{1.37}$$

diverges for all $h \neq 0$. Nevertheless, existence of the limits (1.35) happens to be a sufficient condition for the series (1.37) to be asymptotic [44]. That is, for all

[42] Privman V., Schulman L.S., *Analytic Properties of Thermodynamic Functions at First Order Phase Transitions*, J. Phys. A: Math. Gen. **15**, 231-238, (1982). Privman V., Schulman L.S., *Analytic Continuation at First Order Phase Transitions*, J. Stat. Phys. **29**, 2, (1982).

[43] Isakov S.N., *Non-analytic Features of the First Order Phase Transition in the Ising Model*, Commun. Math. Phys. **95**, 427-443, (1984).

[44] Remmert R., *Theory of Complex Functions*, Springer, (1990), p. 296.

integer n,

$$\lim_{h\searrow 0^+}\left[\mathfrak{p}(h)-\sum_{k=0}^{n}\frac{1}{k!}\mathfrak{p}^{(k)}(0^+)h^k\right]h^{-n}=0\,. \tag{1.38}$$

The structure of the pressure is thus the following: all the limits $\lim_{h\searrow 0^+}\mathfrak{p}^{(k)}(h)$ exist, but allow expressing $\mathfrak{p}$ only as an asymptotic, non convergent expansion at $h=0$. Actually, it can also be shown that this asymptotic expansion is Borel-summable (see Chapter 8).

We saw, in the droplet model, that the precise nature of the singularity was obtained by studying the pressure at $z=1$, i.e. at the boundary of the region of analyticity. Isakov followed the same route and investigated precisely the behaviour of large systems in a neighbourhood of $h=0$. His technique for obtaining the bounds (1.36) is inspired by the droplet mechanism; we will first give a brief description of its main steps, and then present our results in Section 1.2.4.

The Mechanism

The problem we face when studying a model more realistic than those considered by Fisher et al. is that the common representation of the pressure, like the expansion of Mayer, is usually an infinite alternated sum. (We should keep in mind the alternating property $(-1)^{l+1}b_l^0\geq 0$ of the cluster coefficients.) This "sign problem" does not allow a rough estimate of $\mathfrak{p}^{(k)}(0^+)$ by simple determination of the dominant term, as in the droplet model.

What Isakov did to overcome this difficulty, is to work in a large finite box $\Lambda\subset\mathbf{Z}^d$ with boundary condition $+$, and to introduce a new representation for the pressure $\mathfrak{p}^+_\Lambda$, which is defined by

$$\mathfrak{p}^+_\Lambda := \frac{1}{\beta|\Lambda|}\log Z^+(\Lambda)\,. \tag{1.39}$$

For a finite box, the pressure $\mathfrak{p}^+_\Lambda$ is always analytic at $h=0$.

Let $\mathcal{C}^+(\Lambda)$ denote the set of *all* possible Peierls contours associated with spin configurations in Λ. Denote the volume of a contour Γ by $V(\Gamma)$. The partition function $Z^+(\Lambda)$ can be represented as a sum over sub-families of $\mathcal{C}^+(\Lambda)$ of pairwise compatible (disjoint) contours. The family $\mathcal{C}^+(\Lambda)$, which is finite, can be ordered in an arbitrary way, $\mathcal{C}^+(\Lambda)=\{\Gamma_1,\Gamma_2,\dots,\Gamma_N\}$. Consider a volume-preserving order, i.e. $V(\Gamma_i)\leq V(\Gamma_j)$ when $i\leq j$. The predecessor of a contour Γ, with respect to the chosen order, is denoted $i(\Gamma)$. Consider, for all $\Gamma\in\mathcal{C}^+(\Lambda)$, the partition function $Z^+_\Gamma(\Lambda)$, in which any contour must be smaller than Γ. We have $Z^+_{\Gamma_N}(\Lambda) = Z^+(\Lambda)$ and $Z^+_{i(\Gamma_1)}(\Lambda) := 1$. The main idea is to write the partition function as a product:

$$Z^+(\Lambda) = Z^+_{\Gamma_1}(\Lambda)\frac{Z^+_{\Gamma_2}(\Lambda)}{Z^+_{\Gamma_1}(\Lambda)}\frac{Z^+_{\Gamma_3}(\Lambda)}{Z^+_{\Gamma_2}(\Lambda)}\cdots\frac{Z^+_{\Gamma_N}(\Lambda)}{Z^+_{\Gamma_{N-1}}(\Lambda)} = \prod_{\Gamma\in\mathcal{C}^+(\Lambda)}\frac{Z^+_\Gamma(\Lambda)}{Z^+_{i(\Gamma)}(\Lambda)}\,. \tag{1.40}$$

With this, the pressure has the form of a *finite* sum,

$$\mathfrak{p}^+_\Lambda = \frac{1}{\beta|\Lambda|}\sum_{\Gamma\in\mathcal{C}^+(\Lambda)} u^+_\Lambda(\Gamma)\,, \tag{1.41}$$

where

$$u^+_\Lambda(\Gamma) := \log\frac{Z^+_\Gamma(\Lambda)}{Z^+_{i(\Gamma)}(\Lambda)} \tag{1.42}$$

The analysis of the phase diagram of the model implies existence of a complex domain $U_\Gamma \ni 0$ in which each function $u^+_\Lambda(\Gamma)$ is analytic in h. The construction of the domains U_Γ can roughly be described as follows. Like in the droplet model, each contour Γ has a surface energy $\|\Gamma\|$ (for the Ising Model, $\|\Gamma\|$ is just the number of dual bonds on the dual lattice) and a volume energy $2hV(\Gamma)$. The domain U_Γ is defined so that the volume energy doesn't exceed the surface energy: for all $h\in U_\Gamma$,

$$2|\mathrm{Re}\,h|V(\Gamma) \le \theta\|\Gamma\|\,, \tag{1.43}$$

where $\theta\in(0,1)$. Then, on U_Γ (which is a strip centered on the imaginary axis $\{\mathrm{Re}\,h=0\}$), the Boltzmann weight of Γ is bounded by

$$|e^{-\beta\|\Gamma\|-2\beta hV(\Gamma)}| \le e^{-\beta\|\Gamma\|}e^{\theta\beta\|\Gamma\|} \le e^{-(1-\theta)\beta\|\Gamma\|}\,. \tag{1.44}$$

This implies stability of the contour for values $h\in U_\Gamma$. Clearly, (1.43) shows that an optimal construction of the domains U_Γ can be done by studying carefully the ratios $\frac{V(\Gamma)}{\|\Gamma\|}$. It can be shown that for $h\in U_\Gamma$ (the symbol $\simeq$ means there exist upper and lower bounds of the type mentioned),

$$u^+_\Lambda(\Gamma) \simeq e^{-\beta\|\Gamma\|-2\beta hV(\Gamma)}\,. \tag{1.45}$$

The derivatives of the functions $u^+_\Lambda(\Gamma)$ are studied with the help of the Cauchy formula:

$$\frac{\mathrm{d}^k}{\mathrm{d}h^k}u^+_\Lambda(\Gamma)\big|_{h=0} = \frac{k!}{2\pi i}\int_C \frac{u^+_\Lambda(\Gamma)(z)}{z^{k+1}}\mathrm{d}z\,, \tag{1.46}$$

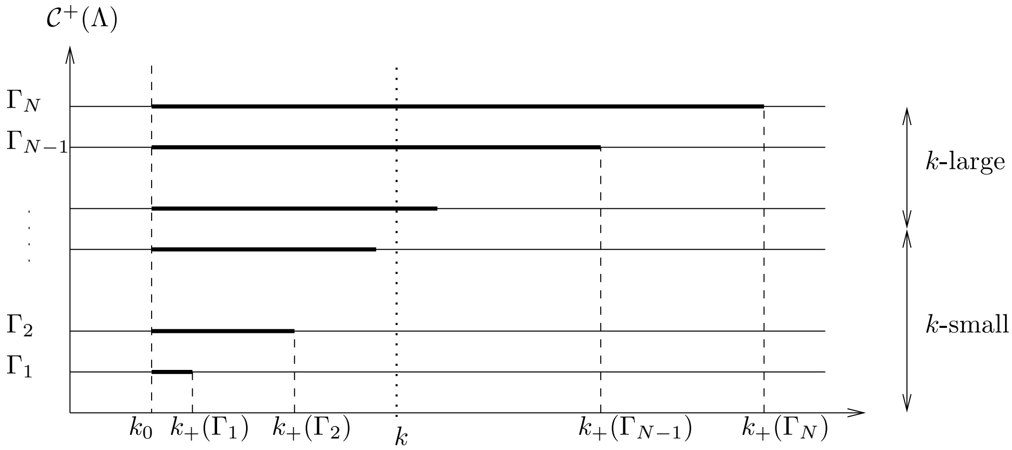

Figure 1.7: The intervals of integers for the derivatives that can be estimated by the method of Isakov: for each Γ_i, the bold interval gives the set of integers k for which precise estimates can be obtained on the k-th derivative of $u^+_\Lambda(\Gamma_i)$ at $h = 0$. We also depicted, at fixed k, the partition of $\{\Gamma_1, \dots, \Gamma_N\}$ into k-small and k-large contours.

where the contour of integration $C \subset U_\Gamma$ can be chosen arbitrarily, encircling the origin. It happens that sharp bounds on the integral can be obtained only when $k \in [k_0, k_+(\Gamma)]$, where k_0 is a fixed constant and $k_+(\Gamma)$ increases with the volume $V(\Gamma)$. Namely, when $k \in [k_0, k_+(\Gamma)]$, the contour of integration can be chosen so as to go through a saddle point giving the dominant contribution to the integral.

For a *fixed* integer k, the family $\mathcal{C}^+(\Gamma)$ then splits in two. The contours Γ for which $[k_0, k_+(\Gamma)] \ni k$ are called *k-large*, those for which $[k_0, k_+(\Gamma)] \not\ni k$ are called *k-small*. This notion is independent of Λ. See Figure 1.7.

$$\mathfrak{p}^+_\Lambda = \frac{1}{\beta|\Lambda|} \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ k-\text{small}}} u^+_\Lambda(\Gamma) + \frac{1}{\beta|\Lambda|} \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ k-\text{large}}} u^+_\Lambda(\Gamma)\,. \tag{1.47}$$

For k-large contours, the saddle point analysis allows a very accurate estimation of the integral in (1.46):

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}h^k} u^+_\Lambda(\Gamma)\big|_{h=0} \simeq (2\beta)^k V(\Gamma)^k e^{-\beta\|\Gamma\|}\,. \tag{1.48}$$

That is, the k-th derivative of the k-large contours all have the *same sign*; they can be treated like in the droplet model, by choosing translates of an appropriate contour maximising the quantity $V(\Gamma)^k e^{-\beta\|\Gamma\|}$. This is nothing but a discrete isoperimetric problem, in which the ratio $\frac{V(\Gamma)}{\|\Gamma\|}$ must be maximised under a k-

dependent constraint. This yields, like in the droplet model,

$$\Big|\frac{1}{\beta|\Lambda|}\sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ k-\text{large}}}\frac{\mathrm{d}^k}{\mathrm{d}h^k}u^+_\Lambda(\Gamma)\big|_{h=0}\Big| \geq A^k k!^{\frac{d}{d-1}}\,. \tag{1.49}$$

This bound is uniform in the size of the box Λ. For the k-small contours, only an upper bound can be obtained, with the Cauchy formula (use the largest possible disc contained in the domains U_Γ):

$$\Big|\frac{1}{\beta|\Lambda|}\sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ k-\text{small}}}\frac{\mathrm{d}^k}{\mathrm{d}h^k}u^+_\Lambda(\Gamma)\big|_{h=0}\Big| \leq B^k k!^{\frac{d}{d-1}}\,. \tag{1.50}$$

At last, the whole point is to show that the contribution from the k-small contours is negligible in comparison with the k-large ones, i.e.

$$A > B\,. \tag{1.51}$$

Typically, texts on Pirogov-Sinai Theory usually provide domains U_Γ whose size is proportional to the inverse of the diameter of the contour Γ, which is insufficient to show that $A > B$.

The main difficulty, in the proof of (1.51), is related to the study of a discrete isoperimetric problem involving the ratios $\frac{V(\Gamma)}{\|\Gamma\|}$. Namely, these appear 1) in the construction of the regions U_Γ, i.e. in B since B depends on the size of the largest disc possible in U_Γ, and 2) in the maximisation of the quantity $V(\Gamma)^k e^{-\beta\|\Gamma\|}$, i.e. in A.

In the Ising Model, the isoperimetric problem can be solved explicitly (the solutions are given by cubes), but in more general models, the surface energy $\|\Gamma\|$ depends on the details of the hamiltonian, and the optimising shapes can be very hard, if not impossible, to find. In a second paper [45], Isakov tried to extend his first theorem to a general class of two phase models. Unfortunately, this could only be done by *assuming* the isoperimetric problems can be solved. We will come back to the discussion of this delicate point at the end of Chapter 4.

The final step is to show that the bounds obtained in a finite volume extend to the thermodynamic limit. That is,

$$\mathfrak{p}^{(k)}(0^+) = \lim_{\Lambda\nearrow\mathbf{Z}^d}\mathfrak{p}^+_\Lambda(0)\,. \tag{1.52}$$

[45] Isakov S.N., *Phase Diagrams and Singularity at the Point of a Phase Transition of the First Kind in Lattice Gas Models*, Teoreticheskaya i Matematicheskaya Fizika, **71**, 3, 426-440, (1987).

This follows from the fact that derivatives of any order are bounded in a neighbourhood of $h = 0$; we will come back to it in details later.

Remark: Although the pressure has no continuation at $h = 0$ in the sense of complex functions, continuations in the sense of real functions always exist, due to the existence of the limits $\mathfrak{p}^{(k)}(0^+)$. For instance, Schonmann and Schlosman [46] proposed a particular C^∞-continuation through the transition point, provided naturally by dynamical considerations, making a link between metastable relaxation and the theory of equilibrium cristal shapes.

1.2.4 New Results on Non-Analyticity

Isakov's papers remain the only rigorous study of non-analyticity at first order phase transition; our results are their natural continuation. They contain two main parts: the first is an extension of Theorem 1.1 to the whole class of two phase models considered in Pirogov-Sinai Theory whereby we show that the Peierls condition is sufficient to guarantee non-analyticity at the transition point; the second result aims at studying how non-analyticity relates to the range of interaction, in the framework of Kac potentials.

Hereafter we briefly present our results; more precise statements are to be found in subsequent chapters, especially concerning Kac potentials (see Chapter 3).

Two Phase Models of Pirogov-Sinai Theory

The framework is the one described in Section 1.1.4. Let H_0 be a hamiltonian with finite range periodic interaction, with two periodic ground states ψ_1, ψ_2, so that the Peierls condition is satisfied. Let $\mathfrak{p} = \mathfrak{p}(\mu)$ denote the pressure of the model with hamiltonian

$$H^\mu = H_0 + \mu H_1 \,, \tag{1.53}$$

where H_1 is a hamiltonian with periodic and finite range interactions that splits the degeneracy of the ground states of H_0. We saw in the Introduction that the main result of Pirogov-Sinai Theory is the existence of a first order phase transition point $\mu^*(\beta)$ such that $\mathfrak{p}$ is analytic in μ on $\{\mu < \mu^*(\beta)\}$ and $\{\mu > \mu^*(\beta)\}$. Our first result is the following (remember Figure 1.3).

Theorem 1.2. *For any two phase model considered by the Pirogov-Sinai Theory, there exists an inverse temperature β^* such that for all $\beta \geq \beta^*$, the pressure has no analytic continuation from $\{\mu < \mu^*(\beta)\}$ to $\{\mu > \mu^*(\beta)\}$ across $\mu^*(\beta)$, or vice versa.*

[46] Schonman R., Schlosman S., *Metastable Kinetics*, Commun. Math. Phys.

The proof of Theorem 1.2, which is the main result of [FP1], can be found in Appendix A.

Kac Potentials

Kac potentials were described in Section (1.1.5). We saw that the Lebowitz-Penrose Theorem gave a closed form to the free energy in the van der Waals Limit $\gamma \searrow 0$ (in which the range of interaction goes to infinity) and justified the Maxwell Construction.

After the van der Waals Limit, the only vestige of the finiteness of the range of interaction is the plateau appearing in the free energy, and the free energy has analytic continuation at condensation/evaporation points. The question arises as to whether this analytic behaviour already holds for small enough γ or if a mechanism similar to the one used by Isakov can be used to show that non-analyticity *persists* for all the smaller values of γ, along the van der Waals Limit $\gamma \searrow 0$. This interesting question was raised by Joel Lebowitz at the Conference *Inhomogeneous Random Systems*, held in Paris, January 2001, and had already appeared, in a less precise form, in the paper of Langer [L].

Consider the lattice gas on $\mathbf{Z}^d$ (see Chapter 2) with Kac potential $K_\gamma(x, y)$, $0 < \gamma < 1$. Denote the free energy of this gas by $f_\gamma = f_\gamma(\rho)$, with particle density $\rho \in (0, 1)$. For simplicity, we consider the potential associated to the step function

$$\varphi(x) := 2^{-d} 1_{\{\|y\| \leq 1\}}(x) , \tag{1.54}$$

where $1_A(x) = 1$ if $x \in A$, zero otherwise.

It is known [47] that for low temperatures and sufficiently small $0 < \gamma < 1$, the free energy f_γ has a phase transition, i.e. is affine on a coexistence plateau $[\rho_g, \rho_l]$. Our contribution is the following.

Theorem 1.3. *There exists an inverse temperature β_0 and $\gamma_0 > 0$ such that for all $\beta \geq \beta_0$ and for all $\gamma \in (0, \gamma_0)$, the following holds: the free energy f_γ is analytic on the gas (resp. liquid) branch $(0, \rho_g)$ (resp. $(\rho_l, 1)$), but has no analytic continuation at $\rho = \rho_g$ (resp. $\rho = \rho_l$) along the path $\rho \nearrow \rho_g$ (resp. $\rho \searrow \rho_l$).*

This result is the content of [FP2]; its proof will be given in details, in the Chapters 2 to 7 of this thesis.

[47] Cassandro M., Presutti E., *Phase Transitions in Ising Systems with Long but Finite Range Interactions*, Mark. Proc. and Rel. Fields **2**, 241-262, (1996), Bovier A., Zahradník M., *The Low-Temperature Phases of Kac-Ising Models*, J. Stat. Phys. **87**, 311-332, (1997).

In other words, as long as the range of interaction is finite ($\gamma > 0$), the free energy has singularities at condensation and evaporation points that block the analytic continuation along the gas and liquid branches; analytic continuation occurs only *after* the van der Waals Limit, as described in the Lebowitz-Penrose Theorem. In particular when $\gamma > 0$, there is no way of obtaining f_γ from a Maxwell Construction, i.e. by taking the convex envelope of an analytic function.

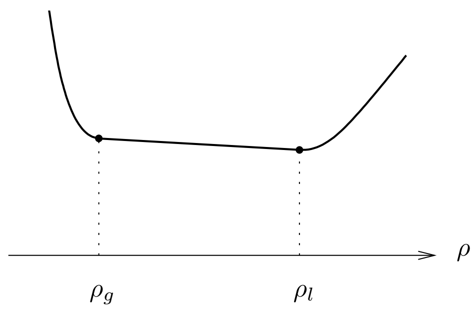

Figure 1.8: The free energy for small $\gamma > 0$, which has singularities at condensation and evaporation points blocking analytic continuation along the paths $\rho \nearrow \rho_g$, $\rho \searrow \rho_l$. Compare with Figure 1.4.

A non-trivial feature of Theorem 1.3 is that β_0 is independent of the scaling parameter γ, which allows studying the van der Waals Limit at fixed temperature. Namely, Theorem 1.2 applies for each choice of $0 < \gamma < 1$, but the range of temperatures for which the result then depends on γ, and shrinks to zero in the van der Waals Limit. The key ingredient for obtaining uniformity in γ, in Theorem 1.3, will be to use a coarse-grained description of the model.

1.3 Overview of the Rest of the Thesis

The rest of the thesis is essentially devoted to the proofs of Theorems 1.2 and 1.3. We have chosen to keep the proof of Theorem 1.2 separate from the rest of the thesis, in Appendix A. The reason for this is that it uses different notations, faithful with those of the book of Sinai [S]. Appendix A can be read independently of the rest of the thesis.

In Chapter 2 we start the rigorous description of the lattice gas with two body potentials $K(i,j)$, of which $K_\gamma(i,j)$ is a special case. Therein we introduce the main thermodynamic potentials and make precise the relationship existing between this lattice gas and the ferromagnet with couplings $J(i,j)$. In particular, we show how all the analyticity properties can be inherited from each other. This allows, in Chapter 3, to reformulate our main result only in terms of the pressure of the Kac ferromagnet (Theorem 3.4).

Chapters 4 to 7 contain the proof of Theorem 3.4; see the end of Chapter 3 for a description of the strategy proof. The stationary phase analysis, which is a crucial ingredient for estimating the derivatives of k-large contours, is given in details in Appendix B. The cluster expansion technique, which is used at several places during the proof, is briefly exposed in Appendix C.

In Chapter 8, we make some concluding remarks and discuss of a few open problems.

Chapter 2

The Simple Lattice Gas

In this chapter we present the lattice description of the liquid-vapor equilibrium, in its simplest form. The properties of the lattice gas will be inherited from those of the Ising ferromagnet, in which symmetry allows to use strong existing results (correlation inequalities, Circle Theorem, etc.). In particular, we show that the analyticity properties can be deduced one from the other.
Our notations will be the following: $p = p(\mu)$ and $f = f(\rho)$ denote the pressure and free energy density of the lattice gas, whereas $\mathfrak{p} = \mathfrak{p}(h)$ and $\mathfrak{f} = \mathfrak{f}(m)$ denote the pressure and free energy density of the ferromagnet. Some general properties of these functions, such as existence or convexity, can be found e.g. in the monograph of Israel [Isr].

2.1 Thermodynamic Potentials

The lattice gas is defined on the lattice $\mathbf{Z}^d$, $d \geq 2$. The distance we use is $d(x,y) := \|x - y\|$, where

$$\|x\| := \max_{1 \leq i \leq d} |x_i| \,. \tag{2.1}$$

This distance will also be used for points of $\mathbb{R}^d$. The letter Λ will always denote a finite subset of $\mathbf{Z}^d$.
The thermodynamic limit will be taken along a sequence of boxes $\Lambda_N = [-N, +N]^d \cap \mathbf{Z}^d$. In the sequel, when $F = F(\Lambda)$, we will use the following notation for the thermodynamic limit (when it exists):

$$\lim_{\Lambda \nearrow \mathbf{Z}^d} F(\Lambda) := \lim_{N \to \infty} F(\Lambda_N) \,. \tag{2.2}$$

The Simple Lattice Gas

At each site of the lattice, a variable ω_i can take two values, 0 or 1. If $\omega_i = 1$, we say that there is a particle at the site i, and when $\omega_i = 0$ we say that the site i is empty. Particles are indistinguishable. The set of particle configurations in a set Λ is denoted $\{0,1\}^\Lambda$. Repulsion at short distance, as in van der Waals Theory, is modelised by the fact that there can exist at most one particle at each site. The

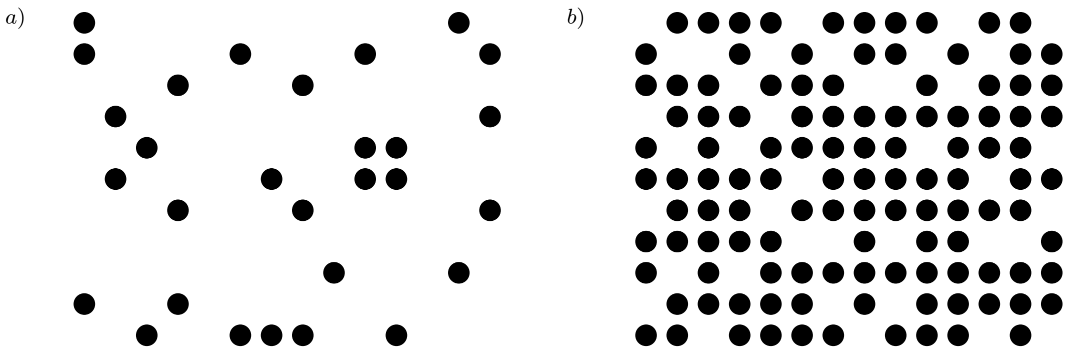

Figure 2.1: Expected configurations of the lattice gas *a*) in the gas phase, with density ρ_g, *b*) in the liquid phase, with density $\rho_l > \rho_g$.

long distance attractive potential between two particles located at sites $i, j \in \mathbf{Z}^d$ is realised with a coupling $K(i,j) \geq 0$, which is assumed to have the symmetry $K(i,j) = K(j,i)$, to be translation invariant: $K(i+x, j+x) = K(i,j)$ for all $x \in \mathbf{Z}^d$, and to be summable in the sense that

$$\hat{K} := \sum_{j \neq 0} K(0,j) < \infty \,. \tag{2.3}$$

The energy of a configuration of particles in Λ, $\omega \in \{0,1\}^\Lambda$, is obtained via the hamiltonian, as follows:

$$H_\Lambda(\omega) := -\sum_{\substack{\{i,j\} \subset \Lambda \\ i \neq j}} K(i,j)\omega_i\omega_j \,. \tag{2.4}$$

When the number of particles in Λ is fixed, $\sum_{i\in\Lambda} \omega_i = N$, the canonical partition function, at inverse temperature $\beta > 0$, is defined by

$$Q^{\mathrm{can}}(\Lambda, N, \beta) := \sum_{\substack{\omega \in \{0,1\}^\Lambda \\ \sum_{i\in\Lambda}\omega_i = N}} e^{-\beta H_\Lambda(\omega)} \,. \tag{2.5}$$

For $N \in [0, |\Lambda|]$, define the free energy in Λ, $f_\Lambda(N, \beta)$, by

$$e^{-\beta f_\Lambda(N,\beta)|\Lambda|} := Q^{\mathrm{can}}(\Lambda, N, \beta) \,. \tag{2.6}$$

Let $\rho \in [0,1]$. For each box Λ, let $N(\Lambda) \in [0, |\Lambda|]$ be such that $\frac{N(\Lambda)}{|\Lambda|} \to \rho$ in the thermodynamic limit. The free energy density is given by

$$f(\rho, \beta) := \lim_{\Lambda \nearrow \mathbf{Z}^d} f_\Lambda(N(\Lambda), \beta) \,, \tag{2.7}$$

and $f(\rho,\beta) := +\infty$ when $\rho \notin [0,1]$. $f(\rho,\beta)$ exists and is convex in ρ. When the number of particles can fluctuate, the grand canonical partition function, at inverse temperature $\beta > 0$ and chemical potential $\mu \in \mathbb{R}$, is defined by

$$Q^{\text{gcan}}(\Lambda,\mu,\beta) := \sum_{N=0}^{|\Lambda|} e^{\beta\mu N} Q^{\text{can}}(\Lambda,N,\beta)\,. \tag{2.8}$$

The pressure in Λ, $p_\Lambda(\mu,\beta)$, is defined by

$$e^{\beta p_\Lambda(\mu,\beta)|\Lambda|} := Q^{\text{gcan}}(\Lambda,\mu,\beta)\,. \tag{2.9}$$

The pressure density is obtained by taking the thermodynamic limit

$$p(\mu,\beta) := \lim_{\Lambda \nearrow \mathbf{Z}^d} p_\Lambda(\mu,\beta)\,. \tag{2.10}$$

This limit exists and is convex in μ. Most often, we will drop β from the notation: $p(\mu,\beta) \equiv p(\mu)$, $f(\rho,\beta) \equiv f(\rho)$. As will be seen, the pressure density can have at most one first order phase transition point, at $\mu^* := -\hat{K}$. The interval $(-\infty,\mu^*]$ is the gas branch and $[\mu^*,+\infty)$ is the liquid branch. When $\mu \neq \mu^*$, p is differentiable with respect to μ, and we define the particle density

$$\rho(\mu) := \frac{\partial p}{\partial \mu}\,. \tag{2.11}$$

At μ^*, only directional derivatives can be defined. Their existence is guaranteed by the convexity of the pressure.

$$\rho_g := \lim_{\epsilon \searrow 0^+} \frac{p(\mu^*) - p(\mu^* - \epsilon)}{\epsilon}\,, \qquad \rho_l := \lim_{\epsilon \searrow 0^+} \frac{p(\mu^* + \epsilon) - p(\mu^*)}{\epsilon}\,. \tag{2.12}$$

At low temperature we expect that $\rho_g < \rho_l$, like in Figure 2.2. The equivalence of ensembles states that the pressure and free energy densities are related one to the other by a Legendre transform:

$$f(\rho) = \sup_\mu \big(\mu\rho - p(\mu)\big)\,, \quad p(\mu) = \sup_\rho \big(\rho\mu - f(\rho)\big)\,. \tag{2.13}$$

At low temperature, $p(\mu)$ and $f(\rho)$ are expected to behave as in Figure 2.2.

The Ferromagnet

At each site of the lattice, a spin variable σ_i can take two values, -1 or $+1$. The set of spin configurations on a finite set Λ is denoted $\Omega_\Lambda = \{\pm 1\}^\Lambda$. A ferromagnetic interaction between two spins located at sites i,j is modelised with a coupling $J(i,j) \geq 0$, which is assumed to have the symmetry $J(i,j) = J(j,i)$,

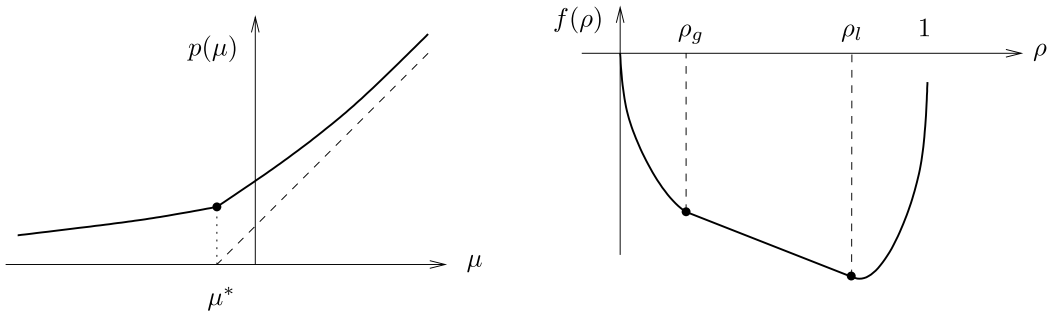

Figure 2.2: The expected graphs of the pressure and free energy of the lattice gas at low temperature.

to be translation invariant: $J(i+x, j+x) = J(i,j)$ for all $x \in \mathbf{Z}^d$, and to be summable in the sense that

$$\hat{J} := \sum_{j \neq 0} J(0,j) < \infty \,. \tag{2.14}$$

For a spin configuration $\sigma \in \Omega_\Lambda$, the hamiltonian is defined by:

$$H_\Lambda^h(\sigma) := -\sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} J(i,j)\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i \,, \tag{2.15}$$

where $h \in \mathbb{R}$ is a magnetic field. When $h = 0$ we have the spin-flip symmetry $H_\Lambda^h(-\sigma) = H_\Lambda^{-h}(\sigma)$. Define

$$\mathcal{M}_\Lambda := \{-|\Lambda|, -|\Lambda|+2, \ldots, |\Lambda|-2, |\Lambda|\} \,. \tag{2.16}$$

Let $M \in \mathcal{M}_\Lambda$. The canonical partition function associated to this ferromagnet, at inverse temperature $\beta > 0$ is defined by

$$Z^{\mathrm{can}}(\Lambda, M, \beta) := \sum_{\substack{\sigma\in\Omega_\Lambda \\ \sum_{i\in\Lambda}\sigma_i = M}} e^{-\beta H_\Lambda^0(\sigma)} \,. \tag{2.17}$$

For $M \in \mathcal{M}_\Lambda$, define the free energy in Λ, $\mathfrak{f}_\Lambda(M,\beta)$, by

$$e^{-\beta \mathfrak{f}_\Lambda(M,\beta)|\Lambda|} := Z^{\mathrm{can}}(\Lambda, M, \beta) \,. \tag{2.18}$$

Let $m \in [-1,+1]$. For each cube Λ, let $M(\Lambda) \in \mathcal{M}_\Lambda$ be such that $\frac{M(\Lambda)}{|\Lambda|} \to m$ in the thermodynamic limit. The free energy density is defined, for $m \in [-1,+1]$, by

$$\mathfrak{f}(m,\beta) := \lim_{\Lambda \nearrow \mathbf{Z}^d} \mathfrak{f}_\Lambda(M(\Lambda), \beta) \,, \tag{2.19}$$

and $\mathfrak{f}(m,\beta) := +\infty$ when $m \notin [-1,+1]$. The free energy density exists and is convex in m. The (grand canonical) partition function associated to this ferromagnet, at inverse temperature $\beta > 0$ and magnetic field $h \in \mathbb{R}$, is defined by

$$Z(\Lambda, h, \beta) := \sum_{\sigma \in \{\pm 1\}^\Lambda} e^{-\beta H_\Lambda^h(\sigma)} = \sum_{M \in \mathcal{M}_\Lambda} e^{\beta h M} Z^{\mathrm{can}}(\Lambda, M, \beta) \,. \tag{2.20}$$

The pressure in Λ, $\mathfrak{p}_\Lambda(h,\beta)$, is defined by

$$e^{\beta \mathfrak{p}_\Lambda(h,\beta)|\Lambda|} := Z(\Lambda, h, \beta) \,, \tag{2.21}$$

and the pressure density is obtained by taking the thermodynamic limit

$$\mathfrak{p}(h,\beta) := \lim_{\Lambda \nearrow \mathbf{Z}^d} \mathfrak{p}_\Lambda(h,\beta) \,. \tag{2.22}$$

The pressure density exists and is convex in h. Usually, we will drop β from the notations. By the spin-flip symmetry, we have $\mathfrak{p}(-h) = \mathfrak{p}(h)$. By the Theorem of Yang and Lee (see Theorem 2.2 hereafter), $\mathfrak{p}$ can have a phase transition only at $h = 0$. When $h \neq 0$, the pressure is differentiable with respect to h, and we define the magnetisation

$$m(h) := \frac{\partial \mathfrak{p}}{\partial h}(h) \,. \tag{2.23}$$

At $h = 0$, convexity guarantees the existence of the directional derivative:

$$m^* = m^*(\beta) := \lim_{\epsilon \searrow 0^+} \frac{\mathfrak{p}(\epsilon) - \mathfrak{p}(0)}{\epsilon} \,. \tag{2.24}$$

By the equivalence of ensembles, we have

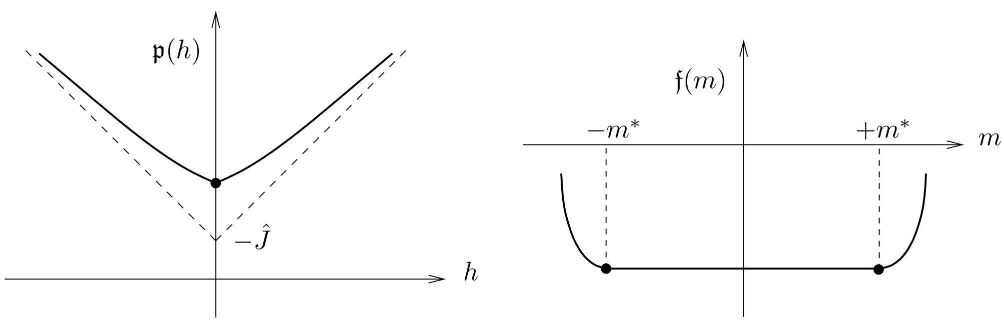

Figure 2.3: The thermodynamic potentials of the ferromagnet when there is a phase transition at $h = 0$. $\mathfrak{f}(m)$ is the Legendre transform of $\mathfrak{p}(h)$.

$$\mathfrak{f}(m) = \sup_h \big(hm - \mathfrak{p}(h)\big) \,, \quad \mathfrak{p}(h) = \sup_m \big(mh - \mathfrak{f}(m)\big) \,. \tag{2.25}$$

We have represented the expected behaviour of $\mathfrak{p}(h)$ and $\mathfrak{f}(m)$ on Figure 2.3. A convenient method, in the study of ferromagnets, is the use of correlation inequalities. For instance, the Griffiths-Kelly-Sherman inequalities [GKS] imply

Lemma 2.1. *For all $\beta > 0$,*

$$\frac{\partial m}{\partial h}(h) > 0 \quad \forall\, h \neq 0\,, \quad \liminf_{h \searrow 0^+} \frac{\partial m}{\partial h}(h) > 0\,. \tag{2.26}$$

Proof. Let $\langle \cdot \rangle_{\Lambda,h}$ denote the Gibbs state in the finite volume Λ with free boundary conditions. An elementary computation and the second GKS inequality allow to bound

$$\frac{\partial^2 \mathfrak{p}_\Lambda}{\partial h^2} = \frac{\beta}{|\Lambda|} \sum_{i,j\in\Lambda} \langle \sigma_i \sigma_j \rangle_{\Lambda,h} - \langle \sigma_i \rangle_{\Lambda,h} \langle \sigma_j \rangle_{\Lambda,h} \geq \frac{\beta}{|\Lambda|} \sum_{i\in\Lambda} (1 - \langle \sigma_i \rangle_{\Lambda,h}^2)\,. \tag{2.27}$$

We kept only the indices $i = j$. Using the first GKS inequality gives, for all $i \in \Lambda$ (apply a magnetic field $h \nearrow +\infty$ on each $j \neq i$),

$$\langle \sigma_i \rangle_{\Lambda,h} \leq \langle \sigma_i \rangle_{\{i\},h+\hat{J}} = \tanh(\beta(h + \hat{J}))\,. \tag{2.28}$$

This implies the following lower bound, uniform in Λ:

$$\frac{\partial^2 \mathfrak{p}_\Lambda}{\partial h^2} \geq \beta\big(1 - \tanh(\beta(h + \hat{J}))^2\big)\,. \tag{2.29}$$

By the Theorem of Yang and Lee (see Theorem 2.2 hereafter), the derivatives and the thermodynamic limit can be exchanged outside $h = 0$, yielding

$$\frac{\partial m}{\partial h} = \frac{\partial^2 \mathfrak{p}}{\partial h^2} = \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{\partial^2 \mathfrak{p}_\Lambda}{\partial h^2} \geq \big(1 - \tanh(\beta(h + \hat{J}))^2\big)\,. \tag{2.30}$$

This in turn implies (2.26). □

The simplest coupling for which the ferromagnet has a phase transition is the Ising model, whose couplings are defined by

$$J(i,j) := \begin{cases} 1 & \text{if } i,j \text{ are nearest neighbours,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.31}$$

In this case, we have the following result due to Peierls [Pei], Griffiths [G] and Dobrushin [Dob].

Theorem 2.1. *If β is large enough, the Ising model has a first order phase transition at $h = 0$, characterised by* spontaneous magnetisation*: $m^*(\beta) > 0$. As a consequence, the* critical temperature *is well defined:*

$$\beta_c := \inf\{\beta > 0 : m^*(\beta) > 0\}\,. \tag{2.32}$$

Lattice Gas vs. Ferromagnet

We go back to the general case $J(i,j) \geq 0$. We map each spin configuration to a particle configuration as follows:

$$\omega_i := \frac{\sigma_i + 1}{2} \tag{2.33}$$

A spin $+1$ thus corresponds to a site containing a particle and a spin -1 corresponds to a vacant site. The thermodynamic properties of the lattice gas and of the ferromagnet can be studied one from the other using the following proposition, which first appeared, in a slitghly different form, in the second paper of Yang and Lee [YL].

Proposition 2.1. *Consider the pressure density $\mathfrak{p} = \mathfrak{p}(h)$ of the ferromagnet with coupling constants $J(i,j)$. Define the function*

$$h(\mu) := \frac{1}{2}(\mu + 4\hat{J}) \,. \tag{2.34}$$

Then the pressure $p = p(\mu)$ of the simple lattice gas with $K(i,j) = 4J(i,j)$ can be obtained by the following identity:

$$p(\mu) = \mathfrak{p}\big(h(\mu)\big) + h(\mu) - \hat{J} \,. \tag{2.35}$$

In the sequel we will always assume that the gas under consideration is related to a given ferromagnet via $K(i,j) = 4J(i,j)$. The function $h(\mu)$ can be inverted: $\mu(h) = 2h - 4\hat{J}$. Since the ferromagnet has a possible phase transition at $h = 0$, the lattice gas has a possible phase transition at

$$\mu^* := \mu(0) = -4\hat{J} = -\hat{K} \,. \tag{2.36}$$

Outside the transition point, the particle density of the gas and the magnetisation of the ferromagnet are related by

$$\rho = \frac{\partial p}{\partial \mu} = \frac{\partial \mathfrak{p}}{\partial h}\frac{\partial h}{\partial \mu} + \frac{\partial h}{\partial \mu} = \frac{1+m}{2} \,. \tag{2.37}$$

At the transition point we have

$$\rho_g = \frac{1-m^*}{2} \qquad \rho_l = \frac{1+m^*}{2} \,, \qquad \text{i.e. } \rho_g + \rho_l = 1 \,. \tag{2.38}$$

The GKS inequalities imply, as in (2.26),

$$\frac{\partial \rho}{\partial \mu}(\mu) > 0 \,, \quad \forall \mu \neq \mu^* \,, \quad \liminf_{\mu \searrow \mu^*} \frac{\partial \rho}{\partial \mu}(\mu) > 0 \,. \tag{2.39}$$

We express the pressure as a function of the particle density ρ and as a function of the specific volume $v := \rho^{-1}$. By (2.39), the map $\rho : (\mu^*, +\infty) \to (\rho_l, 1)$ is bijective, and can be inverted: $\xi_l : (\rho_l, 1) \to (\mu^*, +\infty)$. The same can be done on the gas branch, yielding $\xi_g : (0, \rho_g) \to (-\infty, \mu^*)$. Define, for $\rho \in (0, 1)$,

$$\mu(\rho) := \begin{cases} \xi_g(\rho) & \text{if } \rho \in (0, \rho_g)\,, \\ \mu^* & \text{if } \rho \in [\rho_g, \rho_l]\,, \\ \xi_l(\rho) & \text{if } \rho \in (\rho_l, 1)\,. \end{cases} \tag{2.40}$$

The pressure as a function of the particle density,

$$p(\rho) := p(\mu(\rho))\,, \tag{2.41}$$

is depicted on Figure 2.4 a). Let $v_l := \rho_l^{-1}$, $v_g := \rho_g^{-1}$. On Figure 2.4 b), we expressed the pressure as a function of the specific volume $v = \rho^{-1}$:

$$p(v) := p(\mu(v^{-1}))\,. \tag{2.42}$$

We see that each ferromagnet with a phase transition at $h = 0$ provides, via Proposition 2.1, a lattice gas with a liquid-vapor transition.

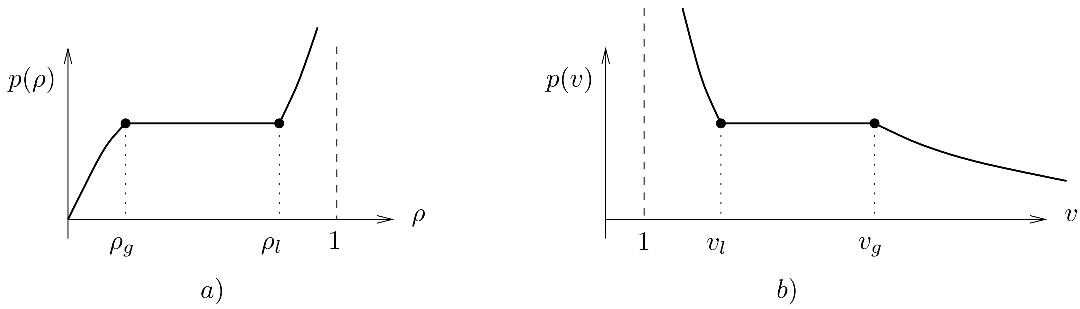

Figure 2.4: The pressure as a function a) of the particle density b) of the specific volume, in the case where there is a phase transition.

It remains to discuss the analyticity properties in the pure phases and at condensation/evaporation points.

2.2 Analyticity Properties. Equivalences

In the present section, we relate the general analyticity properties of the lattice gas to those of the ferromagnet. More precisely, we show that the analyticity properties, of any potential, can be derived from those of the pressure of the ferromagnet.

Analyticity Properties in Pure Phases

The main result on analyticity for the ferromagnet is the Circle Theorem of Yang and Lee [YL], which we already mentionned in Section 1.1.3.

Theorem 2.2 (Yang and Lee, 1952). *We have $Z(\Lambda, h, \beta) \neq 0$ for all Λ and for all $h \in \mathbb{C}$, $\mathrm{Re}\, h \neq 0$. The pressure density $\mathfrak{p} = \mathfrak{p}(h)$ of the ferromagnet is analytic on the domains $\{\mathrm{Re}\, h > 0\}$ and $\{\mathrm{Re}\, h < 0\}$. Moreover, on these domains,*

$$\mathfrak{p}^{(k)}(h) = \lim_{\Lambda \nearrow \mathbf{Z}^d} \mathfrak{p}_\Lambda^{(k)}(h)\,. \tag{2.43}$$

As a consequence, the ferromagnet can have a phase transition only at $h = 0$. In turn, the lattice gas can have a phase transition only at μ^* and is analytic on the liquid $\{\mathrm{Re}\, \mu > \mu^*\}$ and gas branches $\{\mathrm{Re}\, \mu < \mu^*\}$.

Since it is the derivative of the pressure with respect to h, the magnetisation $m : (0, +\infty) \to (m^*, 1)$ is also analytic (the same holds on the branch $(-\infty, 0)$). The following result shows that in a pure phase ($h \neq 0$ or $m \notin [-m^* + m^*]$), the inverse of the magnetisation exists and is also analytic.

Corollary 2.1. *The inverse $h : (m^*, 1) \to (0, +\infty)$ is analytic in a complex neighbourhood of each $m \in (m^*, 1)$. Similarly, the particle density $\rho : (\mu^*, +\infty) \to (\rho_l, 1)$ is analytic in a complex neighbourhood of each $\mu \in (\mu^*, +\infty)$. The inverse $\mu : (\rho_l, 1) \to (\mu^*, +\infty)$ is analytic in a complex neighbourhood of each $\rho \in (\rho_l, 1)$. As a consequence, the pressure $p = p(\rho)$ (see (2.41)) is analytic on the liquid branch $(\rho_l, 1)$, and $p = p(v)$ (see (2.42)) is analytic on $(1, v_l)$. The same holds on the gas branch.*

The proof follows from the fact that the derivatives $\frac{\partial m}{\partial h}$, $\frac{\partial \rho}{\partial \mu}$ are strictly positive in a pure phase (see (2.26), (2.39)), and from the following theorem, a proof of which can be found in [Rem1], p. 281-282.

Biholomorphic Mapping Theorem. *Let $g : D \to \mathbb{C}$ be analytic and $z_0 \in D$ a point such that $g'(z_0) \neq 0$. Then there exists a domain $V \subset D$ containing z_0, such that the following holds: $V' = g(V)$ is a domain, and the map $g : V \to V'$ has an inverse $g^{-1} : V' \to V$ which is analytic, and which satisfies, for all $\omega \in V'$, $g^{-1\prime}(\omega) = \big(g'(g^{-1}(\omega))\big)^{-1}$.*

Analyticity Properties at the Transition Point

We now give a result which aims at showing that the analyticity properties of all the thermodynamic potentials at condensation/evaporation can be deduced from those of the ferromagnet at $h = 0$. A complete equivalence will be obtained only under a further assumption on the second derivative of the pressure (i.e. the succeptibility), which must remain bounded in a neighbourhood of $h = 0$.

Theorem 2.3. *Consider the following statements.*
1) The pressure of the ferromagnet $\mathfrak{p} = \mathfrak{p}(h)$ has analytic continuation at $h = 0$ along the path $h \searrow 0$ (resp. $h \nearrow 0$).
2) The pressure of the lattice gas $p = p(\mu)$ has analytic continuation at μ^* along the path $\mu \searrow \mu^*$ (resp. $\mu \nearrow \mu^*$).
3) The pressure of the lattice gas $p = p(\rho)$ has analytic continuation at ρ_l along the path $\rho \searrow \rho_l$ (resp. at ρ_g along the path $\rho \nearrow \rho_g$).
4) The pressure of the lattice gas $p = p(v)$ has analytic continuation at v_l along the path $v \nearrow v_l$ (resp. at v_g along the path $v \searrow v_g$).
Then we have the equivalences 1) $\iff$ 2) $\implies$ 3) $\iff$ 4). *Moreover, if $\frac{\partial^2 p}{\partial \mu^2}$ is bounded in a neighbourhood of μ^*, then* 2) $\impliedby$ 3).

Proof. The equivalence 1) $\iff$ 2) follows from the identity (2.35). Assume 2). Then by derivation, $\rho = \rho(\mu)$ has analytic continuation along $\mu \searrow \mu^*$. Then, (2.39) and the biholomorphic mapping theorem imply that the inverse $\mu = \mu(\rho)$ has analytic continuation along $\rho \searrow \rho_l$. This implies that the composition $p(\rho) := p(\mu(\rho))$ has analytic continuation along $\rho \searrow \rho_l$. This shows 2) $\implies$ 3). The equivalence 3) $\iff$ 4) is obtained easily.
Assume 3) holds. Then $\rho \mapsto \frac{\partial p}{\partial \rho}$ has analytic continuation along $\rho \searrow \rho_l$. Using (2.41) gives, for $\rho \in (\rho_l, 1)$,

$$\frac{\partial p}{\partial \rho} = \frac{\partial p}{\partial \mu}\frac{\partial \mu}{\partial \rho}, \quad \text{i.e.} \quad \frac{1}{\rho}\frac{\partial p}{\partial \rho} = \frac{\partial \mu}{\partial \rho}. \tag{2.44}$$

The left hand side is analytic along $\rho \searrow \rho_l$, which implies the right hand side is too. By integration, $\mu = \mu(\rho)$ is also analytic along $\rho \searrow \rho_l$. The last step is to show that the inverse of this map, $\rho = \rho(\mu)$, is analytic in a neighbourhood of μ^*. Indeed, since we are assuming that $\frac{\partial^2 p}{\partial \mu^2}$ is bounded in a neighbourhood of μ^*, we have

$$\frac{\partial \mu}{\partial \rho}(\rho_l) = \lim_{\rho \searrow \rho_l} \frac{\partial \mu}{\partial \rho} = \lim_{\mu \searrow \mu^*} \left(\frac{\partial \rho}{\partial \mu}\right)^{-1} = \lim_{\mu \searrow \mu^*} \left(\frac{\partial^2 p}{\partial \mu^2}\right)^{-1} \neq 0\,. \tag{2.45}$$

By the biholomorphic mapping theorem, this shows that $\rho = \rho(\mu)$ is analytic in a neighbourhood of μ^*. By integration, $p = p(\mu)$ is analytic in a neighbourhood of μ^*. □

Assume $\frac{\partial^2 \mathfrak{p}}{\partial h^2}$ is bounded near $h = 0$. By Theorem 2.3, non-analyticity of the ferromagnet at $h = 0$ implies non-analyticity of the simple gas at condensation and evaporation points, in any of the variables μ, ρ, v.

Analyticity of the Legendre Transforms

Analyticity of pressure densities also imply analyticity of free energy densities, which are obtained by a Legendre transform (see (2.13) and (2.25)).

Theorem 2.4. *The free energy $\mathfrak{f} = \mathfrak{f}(m)$ is analytic in a complex neighbourhood of each $m \in (-1, -m^*) \cup (+m^*, +1)$, and has analytic continuation along $m \searrow m^*$ if $\mathfrak{p}$ has analytic continuation along $h \searrow 0$. Moreover, if $\frac{\partial^2 \mathfrak{p}}{\partial h^2}$ is bounded in a neighbourhood of $h = 0$, then analytic continuation of $\mathfrak{f}$ along $m \searrow m^*$ implies analytic continuation of $\mathfrak{p}$ along $h \searrow 0$.*
Similar results hold for the free energy and pressure densities of the lattice gas.

Proof. Consider the identities (2.25). For $m \geq 0$ we have

$$\mathfrak{f}(m) = h(m)m - \mathfrak{p}(h(m))\,, \tag{2.46}$$

where $h(m)$ is the unique solution of $m = m(h) (= \mathfrak{p}'(h))$. If $\mathfrak{p} = \mathfrak{p}(h)$ has analytic continuation at $h = 0$ then so does $m(h)$. By (2.26) and the biholomorphic function theorem, this implies that $h = h(m)$ has analytic continuation at m^*. By (2.46), $\mathfrak{f}(m)$ is a composition of analytic maps, i.e. analytic.
Inversely, if $\mathfrak{f}(m)$ has analytic continuation at m^*, then so does $\mathfrak{f}'(m) = h(m)$. Using the biholomorphic mapping theorem and the assumption that the second derivative of the pressure is bounded near $h = 0$, we compute

$$h'(m^*) = \lim_{m \searrow m^*} h'(m) = \lim_{h \searrow 0} m'(h)^{-1} = \lim_{h \searrow 0} \left(\frac{\partial^2 \mathfrak{p}}{\partial h^2} \right)^{-1} \neq 0. \tag{2.47}$$

This implies that $m(h)$ and, in turn, $\mathfrak{p}(h)$, have analytic continuation at $h = 0$. □

Chapter 3

Kac Potentials

Kac potentials, which we already described briefly in the Introduction, are a particular case of the potentials described in the previous chapter. Their main characteristic is to provide, in the van der Waals Limit, a rigorous justification of the Maxwell Construction.

One of our main results, Theorem 1.3, was formulated in terms of the simple lattice gas. In Chapter 2, we showed that all the analyticity properties of the lattice gas can be deduced from those of the associated ferromagnet. Therefore, the aim of this chapter is to give a stronger version of Theorem 1.3, in terms of the ferromagnet.

The notations are the same as those of Chapter 2: at each site $i \in \mathbf{Z}^d$, $d \geq 2$, lives a spin $\sigma_i \in \{\pm 1\}$. The configuration space is $\Omega = \{\pm 1\}^{\mathbf{Z}^d}$. For any finite set $\Lambda \subset \mathbf{Z}^d$, $\Omega_\Lambda = \{\pm 1\}^\Lambda$. Let $J : \mathbb{R}^d \to \mathbb{R}^+$ be supported by the cube $\{y \in \mathbb{R}^d : \|y\| \leq 1\} = [-1, +1]^d$, so that

$$\int J(x)\mathrm{d}x = 1\,. \tag{3.1}$$

Recall that Kac potentials are defined with the help of a scaling parameter $0 < \gamma < 1$ that allows to tune the range of interaction:

$$J_\gamma(i,j) := c_\gamma \gamma^d J(\gamma(i-j))\,, \tag{3.2}$$

where c_γ is defined in such a way that

$$\hat{J} = \sum_{j \neq 0} J_\gamma(0,j) \equiv 1\,. \tag{3.3}$$

It is easy to see that (3.1) implies $\lim_{\gamma \searrow 0} c_\gamma = 1$. Since $J_\gamma(i,j) = 0$ if $d(i,j) > \gamma^{-1}$, we call $R := \gamma^{-1}$ the range of the interaction.

Convention: *Unless stated explicitly, R will always denote the range of interaction, i.e. γ^{-1}.*

For a finite volume Λ, $\sigma \in \Omega_\Lambda$, the Kac hamiltonian is defined by

$$H^h_\Lambda(\sigma) = -\sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} J_\gamma(i,j)\sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i\,, \tag{3.4}$$

where $h \in \mathbb{R}$ is the magnetic field. The free energy and pressure density associated to this ferromagnet are defined as in Chapter 2 and are denoted

$$\mathfrak{f}_\gamma = \mathfrak{f}_\gamma(m)\,, \quad \mathfrak{p}_\gamma = \mathfrak{p}_\gamma(h)\,. \tag{3.5}$$

Most often, we will drop the inverse temperature β from the notations. For all $0 < \gamma < 1$, $\mathfrak{f}_\gamma$ and $\mathfrak{p}_\gamma$ are convex, and by the equivalence of ensembles, each can be obtained from the other by a Legendre transform, as in (2.25).

3.1 The van der Waals Limit

An interesting property of the potentials $\mathfrak{f}_\gamma$ and $\mathfrak{p}_\gamma$ is that they can be given a closed form in the van der Waals Limit $\gamma \searrow 0$, in which the range of $J_\gamma(i,j)$ goes to infinity. Before stating this result, we must remind the main properties of the mean field model.

3.1.1 Thermodynamic Potentials of Mean Field

The mean field (or Curie-Weiss) model is an approximation of long range systems, in which the geometry looses its fundamental role, and in which the thermodynamic potentials can be computed explicitely. For a finite box Λ, consider the mean field hamiltonian

$$H^{MF,h}_\Lambda(\sigma) := -\frac{1}{|\Lambda|}\sum_{\substack{\{i,j\}\subset\Lambda \\ i\neq j}} \sigma_i\sigma_j - h\sum_{i\in\Lambda}\sigma_i\,. \tag{3.6}$$

This model is not realistic since the coupling constant depends on the volume of the system. In particular, the results of Chapter 2 don't apply. Nevertheless, the simple form of the hamiltonian shows that the canonical partition function can be computed explicitely ($M \in \mathcal{M}_\Lambda$):

$$Z^{MF}(\Lambda, M, \beta) = \sum_{\substack{\sigma\in\Omega_\Lambda \\ \sum_{i\in\Lambda}\sigma_i = M}} e^{-\beta H^{MF,0}_\Lambda(\sigma)} = \binom{|\Lambda|}{\frac{|\Lambda|+M}{2}} e^{\beta\frac{M^2}{2|\Lambda|}} e^{-\beta}\,. \tag{3.7}$$

The free energy density $\mathfrak{f}_{MF}(m)$, $m \in [-1, +1]$, can then be easily computed:

$$\mathfrak{f}_{MF}(m) = -\frac{1}{2}m^2 - \frac{1}{\beta}I(m)\,, \tag{3.8}$$

where

$$I(m) := -\frac{1-m}{2}\log\frac{1-m}{2} - \frac{1+m}{2}\log\frac{1+m}{2}\,. \tag{3.9}$$

There are two competing terms in the mean field free energy $\mathfrak{f}_{MF}$. The first, $-\frac{1}{2}m^2$, is concave. The second,$-\frac{1}{\beta}I(m)$, is convex and depends on β. It is easy to see that $\beta = 1$ is a threshold called the critical temperature of mean field: for $\beta \leq 1$ the free energy $\mathfrak{f}_{MF}$ is strictly convex, but for $\beta > 1$, $\mathfrak{f}_{MF}$ is non-convex, and has two minima $\pm m^*(\beta)$, where $m^*(\beta)$ is the positive solution of

$$m = \tanh(\beta m)\,. \tag{3.10}$$

In the same way, the pressure density $\mathfrak{p}_{MF}(h)$ can be computed and yields

$$\mathfrak{p}_{MF}(h) = \sup_m \big(hm - \mathfrak{f}_{MF}(m)\big)\,. \tag{3.11}$$

The pressure $\mathfrak{p}_{MF}(h)$ is convex. It is the Legendre transform of the free energy, but the free energy is *not* the Legendre transform of the pressure: the equivalence of ensembles doesn't hold in the mean field model.

It must be noted that the thermodynamic potentials of mean field behave analytically at their transition points. Remember that this had been used by Temperley and Katsura in their discussion of the analyticity properties at condensation. Katsura had then conjectured that the mean field behaviour holds in general, i.e. also for finite range models.

3.1.2 The Lebowitz-Penrose Theorem

For all $0 < \gamma < 1$ the functions $\mathfrak{f}_\gamma, \mathfrak{p}_\gamma$ are both convex, and equivalence of ensemble holds. Since a pointwise limit of convex functions is convex, they remain convex in the van der Waals Limit, and we expect them to be related, in some way, to the mean field potentials $\mathfrak{f}_{MF}, \mathfrak{p}_{MF}$.

Let $f = f(x)$ be a real function. The convex envelope of f, $\mathrm{CE}\, f(x)$ denotes the largest convex function smaller than f. In the present setting, the Lebowitz-Penrose Theorem [LP] is the following.

Theorem 3.1. *In the van der Waals Limit,*

$$\mathfrak{f}_\gamma(m) \longrightarrow \mathrm{CE}\,\mathfrak{f}_{MF}(m)\,, \tag{3.12}$$

$$\mathfrak{p}_\gamma(h) \longrightarrow \mathfrak{p}_{MF}(h)\,. \tag{3.13}$$

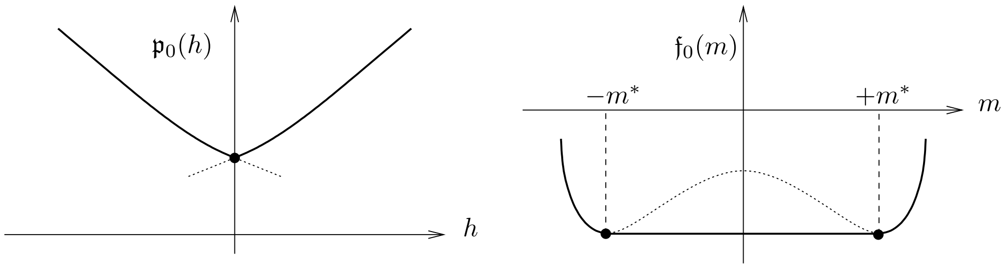

Figure 3.1: The pressure $\mathfrak{p}_0$ and free energy $\mathfrak{f}_0$ when $\beta > 1$. The dotted lines are the analytic continuations provided by mean field.

See the monograph of Presutti [Pr] for a simple and elegant proof of this result. Define $\mathfrak{f}_0(m) := \lim_{\gamma \searrow 0} \mathfrak{f}_\gamma(m)$ and $\mathfrak{p}_0(h) := \lim_{\gamma \searrow 0} \mathfrak{p}_\gamma(h)$. When $\beta > 1$, the graph of $\mathfrak{f}_0$ is flat on the interval $[-m^*(\beta), +m^*(\beta)]$, and $\mathfrak{p}_0$ has a kink at $h = 0$. We have represented $\mathfrak{f}_0$ and $\mathfrak{p}_0$ on Figure 3.1.

A consequence of the Theorem of Lebowitz-Penrose is that after the van der Waals Limit, the analyticity properties are the same as in the van der Waals-Maxwell Theory.

Corollary 3.1. *When $\beta > 1$, the limit free energy $\mathfrak{f}_0$ is analytic everywhere except at $\pm m^*(\beta)$, and has analytic continuations along the (real) paths $m \nearrow -m^*(\beta)$, $m \searrow +m^*(\beta)$. The unique analytic continuation is given by the mean field free energy $\mathfrak{f}_{MF}$. Similarly, the limit pressure $\mathfrak{p}_0$ has analytic continuation at $h = 0$.*

3.2 Long But Finite Range Interactions

The Lebowitz-Penrose Theorem allows to show that in the van der Waals Limit,
1) the system has a phase transition,
2) the analyticity properties are completely determined by those of the mean field model.

With regard to 1), a natural question is to know whether the system has a phase transition also for small $\gamma > 0$, i.e. *before* the van der Waals Limit. As was noted by Presutti in [Pr], it could be that $\mathfrak{f}_\gamma$ is *strictly* convex for all $\gamma > 0$ and $\mathfrak{f}_0$ is flat on $[-m^*(\beta), +m^*(\beta)]$. Indeed, strictly convex functions can approximate arbitrarily well straight segments. It was shown by Cassandro-Presutti [CP] and Bovier-Zahradník [BZ1] that the system does indeed exhibit a phase transition for small enough $\gamma > 0$:

Theorem 3.2. *For all $\beta > 1$ there exists $\gamma(\beta) > 0$ such that for all $\gamma \in (0, \gamma(\beta))$, the free energy $\mathfrak{f}_\gamma$ is flat on some interval $[-m^*(\beta,\gamma), +m^*(\beta,\gamma)]$, i.e. the pressure $\mathfrak{p}_\gamma$ has a discontinuous derivative at $h = 0$.*

With regard to 2), we will now see that the situation is very different when $\gamma > 0$.

3.2.1 Non-Analyticity for $\gamma > 0$

Concerning 2) and in comparison with Theorem 3.2, we are naturally led to the following question: can it be shown that for small enough γ, $\mathfrak{f}_\gamma$ has analytic continuation at $\pm m^*(\beta,\gamma)$? Our result answers negatively to this question, in the case where the function $J(\cdot)$ is the step function

$$J(x) = 2^{-d} 1_{\{\|y\| \leq 1\}}(x) \,. \tag{3.14}$$

Theorem 3.3. *There exists β_0 and $\gamma_0 > 0$ such that for all $\beta \geq \beta_0$, $\gamma \in (0, \gamma_0)$, $\mathfrak{f}_\gamma$ is analytic everywhere except at $\pm m^*(\beta,\gamma)$, and has no analytic continuation along the paths $m \nearrow -m^*(\beta,\gamma)$, $m \searrow +m^*(\beta,\gamma)$.*

Notice that β_0 is *uniform* in the scaling parameter γ: Theorem 3.3 confirms the ideas of Andreev, Fisher and Langer and invalidates the conjecture of Katsura: the free energy has no analytic continuation as long as the range of interaction is finite.

As we saw in Chapter 2, the analyticity properties of the free energy can be obtained from those of the pressure, which is also easier to handle since there is no constraint on the magnetisation. The following theorem is a complete description of the analyticity properties of the pressure density at $h = 0$, again in the case where $J(\cdot)$ is the step function of (3.14). The presence of the symmetry $\mathfrak{p}_\gamma(-h) = \mathfrak{p}_\gamma(h)$ implies that we need only consider analyticity along the path $h \searrow 0$.

Theorem 3.4. *There exists β_0, $\gamma_0 > 0$ and a constant $C_r > 0$ such that for all $\beta \geq \beta_0$, $\gamma \in (0, \gamma_0)$, the following holds:*

1) The limits $\mathfrak{p}^{(k)}(0^+) = \lim_{h \searrow 0} \mathfrak{p}_\gamma^{(k)}(h)$ exist for all $k \in \mathbb{N}$. Moreover, there exists a constant $C_+ > 0$ such that for all $k \in \mathbb{N}$,

$$\sup_{0 < \mathrm{Re}\, h \leq \epsilon} |\mathfrak{p}_\gamma^{(k)}(h)| \leq \left(C_+ \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\right)^k k!^{\frac{d}{d-1}} + C_r^k k! \tag{3.15}$$

where $\epsilon > 0$.
2) The pressure has no analytic continuation at $h = 0$. More precisely, there exists $C_- > 0$ and an unbounded increasing sequence of integers $k_1, k_2, \dots$ such that for all $k \in \{k_1, k_2, \dots\}$,

$$|\mathfrak{p}_\gamma^{(k)}(0^+)| \geq \left(C_- \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\right)^k k!^{\frac{d}{d-1}} - C_r^k k! \,. \tag{3.16}$$

Part 1) of Theorem 3.4 implies that the Taylor series of the pressure at $h = 0$ exists, part 2) shows that it always diverges. Notice that the lower bound (3.16) becomes irrelevant when $\gamma \searrow 0$. Moreover, we should mention that each integer k_i of the sequence $k_2, k_2, \ldots$ depends on γ, and $\lim_{\gamma \searrow 0} k_i = +\infty$: information about non-analyticity is lost in the van der Waals Limit. Since we know from the Lebowitz-Penrose Theorem, and its corollary that $\mathfrak{p}_\gamma$ converges, when $\gamma \searrow 0$, to a function that is is analytic at $h = 0$, it is worthwhile trying to see if this emergence of analyticity can be detected for very small $\gamma > 0$. Considering the upper bound (3.15), it easy to show the following

Corollary 3.2. *There exists $C = C(\beta)$ such that for small values of k, i.e. for $k \leq \gamma^{-d}$, we have the upper bound*

$$\sup_{0<\mathrm{Re}\, h \leq \epsilon} |\mathfrak{p}_\gamma^{(k)}(h)| \leq C^k k! \,, \tag{3.17}$$

This shows that a close inspection of the derivatives of the pressure reveals how analyticity starts to manifest when γ approaches 0. These different behaviours are illustrated on Figure 3.2.

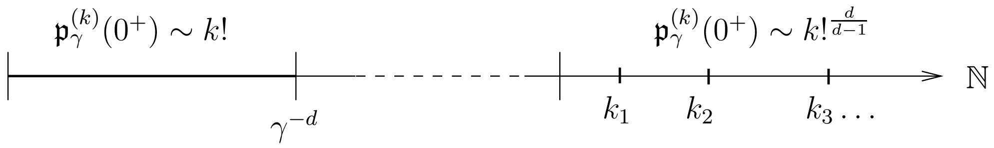

Figure 3.2: The derivatives of the pressure $\mathfrak{p}_\gamma$ at $h = 0$, for small $\gamma > 0$. The first ones ($k \leq \gamma^{-d}$) behave like those of an analytic function, but the large ones reveal the non-analytic feature of the singularity.

By Theorem 3.4, the pressure $\mathfrak{p}_\gamma$ is non-analytic at $h = 0$. A consequence of the upper bound (3.15) is that

$$\sup_{h \neq 0} |\mathfrak{p}_\gamma^{(2)}(h)| < \infty \,. \tag{3.18}$$

By Theorem 2.4, this shows Theorem 3.3.

Consequences for the Lattice Gas. Consider the lattice gas with pair interaction defined by

$$K_\gamma(i,j) := 4J_\gamma(i,j) \,, \tag{3.19}$$

with pressure $p_\gamma = p_\gamma(\mu)$ and free energy $f_\gamma = f_\gamma(\rho)$. By Proposition 2.1, p_γ is related to the pressure of the ferromagnet by

$$p_\gamma(\mu) = \mathfrak{p}_\gamma(h(\mu)) + h(\mu) - \hat{J} \,, \tag{3.20}$$

where $h(\mu) = \frac{1}{2}(\mu + 4\hat{J})$, and $\hat{J} = 1$. As a consequence, the second derivative of p_γ equals

$$\frac{\partial^2 p_\gamma}{\partial \mu^2} = \frac{1}{4}\frac{\partial^2 \mathfrak{p}_\gamma}{\partial h^2}\,. \tag{3.21}$$

Then (3.18) implies that the second derivative of p_γ is bounded near μ^*. By Theorem 3.4 and by the results of Chapter 2, we conclude that

Corollary 3.3. *For all $\beta \geq \beta_0$ and $\gamma \in (0, \gamma_0)$, the pressure p_γ of the lattice gas has no analytic continuation at condensation/evaporation points, in either of the variables μ (chemical potential), ρ (particle density), or v (specific volume). The same holds for the free energy f_γ with respect to the particle density ρ.*

3.2.2 Strategy for the proof of Theorem 3.4.

To show Theorem 3.4, we will first construct the phase diagram of the Kac Model with a complex magnetic field, at low temperatures, γ small. Then, we adapt the technique of Isakov to obtain lower bounds on the derivatives of the pressure in a finite volume. These two essential steps deserve a few comments.

Phase Diagram in a Complex Magnetic Field. As we saw in the Introduction, phase diagrams of lattice systems can be studied in the general framework of Pirogov-Sinai Theory, which applies when the system under consideration has a finite number of ground states, and for which the unperturbed hamiltonian satisfies the Peierls condition. In our case, the Kac potential has two ground states which are the pure $+$ and pure $-$ configurations, but the Peierls constant (computed with respect to these two ground states) goes to zero when $\gamma \searrow 0$ since in the van der Waals Limit, the interaction between two arbitrary spins vanishes. Therefore, a direct application of Pirogov-Sinai Theory, and, in particular, a direct application of Theorem 1.2, would lead to a range of temperature shrinking to zero in the van der Waals Limit.

Recently, Bovier and Zahradník [BZ2] proposed a systematic method to study spin systems with long but finite range interactions. Their technique allows to study, for instance, the Kac Model with a magnetic field, in a range of temperature that is *uniform in* γ. In their approach, the ground states of Pirogov-Sinai Theory are replaced by *restricted phases*, i.e. by *sets* of configurations. In the $+$-restricted phase, for example, all the points are $+$-correct, i.e. their γ^{-1}-neighbourhood contains a majority of spins $+$. When a point is in neither of the restricted phases, it is in the support of a contour Γ, and it can then be shown that the contours defined in this way satisfy the Peierls condition with a Peierls constant ρ that is *uniform* in γ: $\|\Gamma\| \geq \rho|\Gamma|$ where $\|\Gamma\|$ is the surface energy of Γ. We will expose this in details in Chapter 4.

In Chapter 5 we show that a polymer representation can be obtained for the restricted phases, and that their corresponding free energies behave analytically at $h = 0$.

The phase diagram is constructed in Chapter 6, where we give precise domains in which the partition function can be exponentiated. These domains are made optimal by introducing special isoperimetric constants associated to contours (see (4.56)). Complications arise from the fact that polymers of the restricted phases induce interactions among contours. Besides the definition of the restricted ensembles, our analysis of the phase diagram is independent of the paper [BZ2].

Implementing the Mechanism of Isakov. To implement the mechanism used by Isakov, whose main steps were described briefly in the Introduction, we consider the pressure $\mathfrak{p}^+_{\gamma,\Lambda}$ in a finite box Λ, with a pure +-boundary condition. By introducing an order among the contours inside Λ, the pressure can be written as a *finite* sum:

$$\mathfrak{p}^+_{\gamma,\Lambda} = \frac{1}{\beta|\Lambda|} \log Z^+_r(\Lambda) + \frac{1}{\beta|\Lambda|} \sum_{\Gamma \in \mathcal{C}^+(\Lambda)} u^+_\Lambda(\Gamma)\,, \tag{3.22}$$

The difference with (1.41) is the presence of the restricted partition function $Z^+_r(\Lambda)$, in which the configurations satisfy a local constraint on the magnetisation: contours (large fluctuations) are absent but small fluctuations are allowed. In Chapter 7, we use the analysis of Chapters 4 to 6, to study the derivatives of the functions $u^+_\Lambda(\Gamma)$, using a stationary phase analysis. When Λ is sufficiently large, the contributions to $\mathfrak{p}^{+(k)}_{\gamma,\Lambda}(0)$ are the following: since it is analytic, the restricted phase contributes a factor $C^k_r k!$. Then, a class of contours called k-large gives a contribution of order $k!^{\frac{d}{d-1}}$. The rest of the contours is shown to have a negligible contribution in comparison of the k-large ones. This gives a lower bound

$$|\mathfrak{p}^{+(k)}_{\gamma,\Lambda}(0)| \geq \left(C_- \gamma^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\right)^k k!^{\frac{d}{d-1}} - C^k_r k!\,. \tag{3.23}$$

In the last step of the proof we show that $\lim_\Lambda \mathfrak{p}^{+,(k)}_{\gamma,\Lambda}(0) = \mathfrak{p}^{(k),\leftarrow}_\gamma(0)$, and so (3.23) extends to the thermodynamic limit $\Lambda \nearrow \mathbf{Z}^d$, giving the lower bound (3.16) of Theorem 3.4.

As we said in the introduction, the technique leading to lower bounds of the type (3.23) relies heavily on the treatment of some discrete isoperimetric problem. This will be discussed at the end of Chapter 4.

Chapter 4

Contours and Isoperimetric Constants

Contours are the relevant objects for the description of systems near first order phase transition points: they separate regions of space where the system is in one or the other ground state of the system. Introduced originally by Peierls for the Ising Model, the notion of contour was extended by Pirogov-Sinai [PS], Sinai [S] and Zahradník [Z] for finite range models. Their definition depends on the range of interaction: the larger the range, the thicker the contours. This implies that the entropy of the contours, as well as the inverse temperature above which the theory applies, depends on the range of interaction. For a model with fixed range this is not a nuisance but in our case we want to study the van der Waals Limit at fixed temperature, and the standard definition of contour cannot be used.

The key for obtaining a range of temperature that is uniform in γ is to modify the notion of ground state, and to allow small local deviations of the magnetisation. This was done by Bovier and Zahradník in [BZ2] who introduced the notion of *restricted phase*, which is the key for studying long but finite range systems [1].

Contours will be composed of blocks of side length $l = \nu\gamma^{-1}$, $\nu \geq 2$; they separate regions of different restricted phases. Since the range of interaction is γ^{-1}, the analysis of the restricted phases, once the contours are fixed, will depend on the spins specified on the support of the contours; an important precaution must be taken when defining the contours: they must be sufficiently decoupled from the rest of the system. For this reason, we will introduce the parameter $\tilde{\delta}$ in (4.16).

Contours are defined in Section 4.1. The Peierls condition is obtained in Section 4.3. In Section 4.4 we introduce the fundamental isoperimetric constants associated to the contours.

[1] In a different setting, restricted ensembles were also studied in [BS], [DS] and [BKL], [LMP].

Remark: Our definition of contour will be different from the usual one used to study Kac potentials. For instance in [CP] and [BZ1], contours are defined by comparing the local (empirical) magnetisation to the mean field spontaneous magnetisation. This allows to study the system very close to the critical temperature, by using explicitly the mean field functionals. Unfortunately, this technique hasn't yet been extended to the study of the Kac model with a magnetic field. In our case, the local magnetisation is always compared with ± 1 (rather than the spontaneous magnetisation of mean field $\pm m^*$), and we must therefore work at low temperature, not reaching the whole coexistence regime. Moreover, we need to introduce a complex magnetic field, which definitely rules out the possibility of using the standard techniques existing for Kac models.

4.1 Definition of Contours

We introduce some more notations. We have $d(x,\Lambda) = \inf\{d(x,y) : y \in \Lambda\}$. For $N \geq 1$, define the box $B_N(x) := \{y \in \mathbf{Z}^d : d(x,y) \leq N\}$, and $B_N^\bullet(x) := B_N(x)\backslash\{x\}$. The N-neighbourhood of Λ is

$$[\Lambda]_N := \bigcup_{x\in\Lambda} B_N(x)\,, \tag{4.1}$$

and the boundaries

$$\partial_N^+\Lambda := \{x \in \Lambda^c : d(x,\Lambda) \leq N\}\,, \tag{4.2}$$

$$\partial_N^-\Lambda := \{x \in \Lambda : d(x,\Lambda^c) \leq N\}\,. \tag{4.3}$$

A set Λ is N-connected if for all $x, y \in \Lambda$ there exists a sequence $x_1, x_2, \ldots, x_{n-1}, x_n$ with $x_1 = x$, $x_n = y$, $x_i \in \Lambda$, and $d(x_i, x_{i+1}) \leq N$. If $\sigma_\Lambda \in \Omega_\Lambda$, $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$, we define the concatenation $\sigma_\Lambda\eta_{\Lambda^c} \in \Omega$ in the usual way:

$$(\sigma_\Lambda\eta_{\Lambda^c})_i = \begin{cases} (\sigma_\Lambda)_i & \text{if } i \in \Lambda\,, \\ (\eta_{\Lambda^c})_i & \text{if } i \in \Lambda^c\,. \end{cases} \tag{4.4}$$

We often use the symbol # to denote either of the symbols + or −, or the constant configuration taking the value # at each site of $\mathbf{Z}^d$. Recall the definition of $J_\gamma(i,j)$ in (3.2). We define

$$\phi_{ij}(\sigma_i,\sigma_j) := -\frac{1}{2}J_\gamma(i,j)(\sigma_i\sigma_j - 1)\,. \tag{4.5}$$

Let $\phi_{ij} := \phi_{ij}(+,-)$. The overall interaction strength is the upper bound on the energy of interaction of a single spin with the rest of the system, and equals

$$\sum_{j:j\neq i} \phi_{ij} = \sum_{j:j\neq i} J_\gamma(i,j) = 1\,. \tag{4.6}$$

Correct and Incorrect Points. Relevant functions for the study of nearly constant spin regions are the following (they will appear naturally later when reformulating the hamiltonian):

$$w_{ij}^{\#}(\sigma_i,\sigma_j) := \phi_{ij}(\sigma_i,\sigma_j) - \phi_{ij}(\#,\sigma_j) - \phi_{ij}(\sigma_i,\#)\,. \tag{4.7}$$

Notice that $w_{ij}^{\#}(\#,\sigma_j) = w_{ij}^{\#}(\sigma_i,\#) = 0$. Let $\delta \in (0,1)$, $\sigma \in \Omega$. With regard to the step function J defined in (3.14), we define a point i to be $(\delta,+)$-correct for σ if

$$|B_R^{\bullet}(i) \cap \{j : \sigma_j = -1\}| \leq \tfrac{\delta}{2}|B_R(i)|\,. \tag{4.8}$$

That is, the R-neighbourhood of a $(\delta,+)$-correct point contains a majority of $+$ spins. Although we will always consider the step function, it is often easier to formulate proofs with the help of the functions $w_{ij}^{\#}$, since they will appear naturally later in the re-formulation of the hamiltonian. We thus define the notion of correct/incorrect point in the general case.

Definition 4.1. *Let $\delta \in (0,1)$, $\sigma \in \Omega$, $i \in \mathbf{Z}^d$.*
*1) i is $(\delta,+)$-*correct for *σ if $\sum_{j:j\neq i} |w_{ij}^{+}(-,\sigma_j)| \leq \delta$.*
*2) i is $(\delta,-)$-*correct for *σ if $\sum_{j:j\neq i} |w_{ij}^{-}(+,\sigma_j)| \leq \delta$.*
*3) i is δ-*correct for *σ if it is either $(\delta,+)$- or $(\delta,-)$-correct for σ.*
*4) i is δ-*incorrect for *σ if it is not δ-correct.*

It is easy to see that this definition coincides with (4.8) when J is the step function.
The notion of correctness for a point i depends on the spins in the R-neighbourhood of i but neither on the value of σ_i, nor on the magnetic field. Notice that if $\delta = 0$ this notion of correct point essentially coincides with the one of Zahradník in [Z]. We first show that when δ is small, regions of $(\delta,+)$- and $(\delta,-)$-correct points are distant. In particular, a point i cannot be at the same time $(\delta,+)$- and $(\delta,-)$-correct.

Lemma 4.1. *Let $\delta \in (0,2^{-d})$, $\sigma \in \Omega$, $i \in \mathbf{Z}^d$. Then*
1) If i is $(\delta,+)$-correct, the box $B_R(i)$ contains either $(\delta,+)$-correct, or δ-incorrect points, but no $(\delta,-)$-correct points.
2) If i is $(\delta,-)$-correct, the box $B_R(i)$ contains either $(\delta,-)$-correct, or δ-incorrect points, but no $(\delta,+)$-correct points.

Proof. Suppose i is $(\delta,+)$-correct for σ. Consider $j \in B_R(i)$ and compute

$$\sum_{k:k\neq j} |w_{jk}^{-}(+,\sigma_k)| = \sum_{\substack{k\in B_R^{\bullet}(j)\\ \sigma_k=+1}} 2\phi_{jk} \geq \sum_{\substack{k\in B_R^{\bullet}(j)\cap B_R^{\bullet}(i)\\ \sigma_k=+1}} 2\phi_{jk}\,. \tag{4.9}$$

Using the properties of the function $J(\cdot)$ [2], we can exchange j and i and write

$$\sum_{\substack{k\in B^\bullet_R(j)\cap B^\bullet_R(i)\\ \sigma_k=+1}} 2\phi_{jk} = \sum_{\substack{k\in B^\bullet_R(j)\cap B^\bullet_R(i)\\ \sigma_k=+1}} 2\phi_{ik} = \sum_{\substack{k\neq i\\ \sigma_k=+1}} 2\phi_{ik} - \sum_{\substack{k\notin B^\bullet_R(j)\cap B^\bullet_R(i)\\ \sigma_k=+1}} 2\phi_{ik}\,. \tag{4.10}$$

Using (4.6) and $|B_R(j)\cap B_R(i)| \geq 2^{-d}|B_R(i)|$, this last sum can be bounded by

$$\sum_{\substack{k\notin B^\bullet_R(j)\cap B^\bullet_R(i)\\ \sigma_k=+1}} \phi_{ik} \leq \frac{2^d-1}{2^d}\,. \tag{4.11}$$

Then, since i is $(\delta,+)$-correct for σ,

$$\sum_{\substack{k\neq i\\ \sigma_k=+1}} 2\phi_{ik} = 2 - \sum_{\substack{k\neq i\\ \sigma_k=-1}} 2\phi_{ik} = 2 - \sum_{k:k\neq i} |w^+_{ik}(-,\sigma_k)| \geq 2-\delta\,. \tag{4.12}$$

We thus have the lower bound

$$\sum_{k:k\neq j} |w^-_{jk}(+,\sigma_k)| \geq 2-\delta-2\frac{2^d-1}{2^d} > \delta\,, \tag{4.13}$$

i.e. j cannot be $(\delta,-)$-correct for σ, which finishes the proof. □

In the sequel we will always assume that $\delta\in(0,2^{-d})$ is fixed. The cleaned configuration $\overline{\sigma}\in\Omega$ is defined as follows:

$$\overline{\sigma}_i := \begin{cases} +1 & \text{if } i \text{ is } (\delta,+)\text{-correct for } \sigma\,,\\ -1 & \text{if } i \text{ is } (\delta,-)\text{-correct for } \sigma\,,\\ \sigma_i & \text{if } i \text{ is } \delta\text{-incorrect for } \sigma\,. \end{cases} \tag{4.14}$$

For any set $M\subset \mathbf{Z}^d$, we can always consider the partial cleaning $\sigma_M\overline{\sigma}_{M^c}$ which coincides with σ on M and with $\overline{\sigma}$ on M^c. In the sequel, the cleaning and partial cleaning are always done according to the *original* configuration σ, with a fixed δ. Notice that if a point i is, say, $(\delta,+)$-correct for σ, then the cleaning of σ has the only effect, in the box $B_R(i)$, of changing $-$ spins into $+$ spins (and not $+$ spins into $-$ spins). This is a consequence of Lemma 4.1. We denote by $I_\delta(\sigma)$ the set of δ-incorrect points of the configuration σ. The important property of the cleaning operation is stated in the following lemma.

Lemma 4.2. *Let $M_1\subset M_2$, $\delta'\in(0,\delta]$. Then $I_{\delta'}(\sigma_{M_1}\overline{\sigma}_{M_1^c})\subset I_{\delta'}(\sigma_{M_2}\overline{\sigma}_{M_2^c})$.*

[2] At this point we use the particularity of the step function: ϕ_{jk} is constant on the intersection $B^\bullet_R(j)\cap B^\bullet_R(i)$.

Proof. Let i be a $(\delta', +)$-correct point of $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. Using the fact that $\sigma_{M_1}\overline{\sigma}_{M_1^c}$ and $\sigma_{M_2}\overline{\sigma}_{M_2^c}$ coincide on M_1 and M_2^c, we decompose

$$\sum_{k:k\neq i} |w_{ik}^{+}(-,(\sigma_{M_1}\overline{\sigma}_{M_1^c})_k)| = \sum_{\substack{k:k\neq i\\ k\in M_1\cup M_2^c}} |w_{ik}^{+}(-,(\sigma_{M_2}\overline{\sigma}_{M_2^c})_k)| + \sum_{\substack{k:k\neq i\\ k\in M_2\setminus M_1}} |w_{ik}^{+}(-,\overline{\sigma}_k)|$$

There are at most three possibilities for a point k of the last sum. 1) If k is $(\delta, +)$-correct for σ then $\overline{\sigma}_k = +1$ and so $|w_{ik}^{+}(-,\overline{\sigma}_k)| = 0$. 2) If k is δ-incorrect for σ then $\overline{\sigma}_k = \sigma_k = (\sigma_{M_2}\overline{\sigma}_{M_2^c})_k$. 3) If k is $(\delta, -)$-correct for σ then it is also $(\delta, -)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. By Lemma 4.1, i is not $(\delta, +)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$. This is a contradiction with the fact that i is $(\delta', +)$-correct for $\sigma_{M_2}\overline{\sigma}_{M_2^c}$, so this last possibility for k is excluded.
We can then bound the whole sum by δ'. This shows that i is $(\delta', +)$-correct for $\sigma_{M_1}\overline{\sigma}_{M_1^c}$, and finishes the proof. □

Definition of Contours. Contours are defined on a coarse-grained scale. Consider the partition of $\mathbf{Z}^d$ into disjoint cubes $C^{(l)}$ of side length $l \in \mathbb{N}$, $l > 2R$, whose centers lie on the sites of a square sub-lattice of $\mathbf{Z}^d$. We denote by $C_i^{(l)}$ the unique box of this partition containing the site $i \in \mathbf{Z}^d$. $\mathcal{C}^{(l)}$ will denote the family of all subsets of $\mathbf{Z}^d$ that are unions of boxes $C^{(l)}$. For any set $A \subset \mathbf{Z}^d$, consider the thickening (compare with (4.1))

$$\{A\}_l := \bigcup_{i\in A} C_i^{(l)} . \tag{4.15}$$

In the sequel we consider l such that $l = \nu R$, with a fixed $\nu > 2$.
We will need to decouple contours from the rest of the system. Since interactions are of arbitrary large finite range, we follow [BZ2] and introduce a second parameter $\tilde{\delta} \in (0, \delta)$. This new parameter is crucial; its importance will be seen later, for instance in the proof of the analyticity of the restricted phases (more precisely, in the proof of Lemma 5.2). For each $\sigma \in \Omega$ with $|I_{\tilde{\delta}}(\sigma)| < \infty$, consider the following set:

$$\mathcal{E}(\sigma) := \left\{ M \in \mathcal{C}^{(l)} : M \supset [I_\delta(\sigma)]_R,\, M \supset [I_{\tilde{\delta}}(\sigma_M \overline{\sigma}_{M^c})]_R \right\} . \tag{4.16}$$

First we show that $\mathcal{E}(\sigma)$ is not empty. Consider $M_0 := \{[I_{\tilde{\delta}}(\sigma)]_R\}_l$. If $M_0 = \emptyset$ then $I_{\tilde{\delta}}(\sigma) = I_\delta(\sigma) = \emptyset$ and any subset of $\mathbf{Z}^d$ is in $\mathcal{E}(\sigma)$. So we assume $M_0 \neq \emptyset$. This gives $\mathcal{E}(\sigma) \neq \emptyset$ since $M_0 \in \mathcal{C}^{(l)}$, $M_0 \supset [I_{\tilde{\delta}}(\sigma)]_R \supset [I_\delta(\sigma)]_R$ and $M_0 \supset [I_{\tilde{\delta}}(\sigma)]_R \supset [I_{\tilde{\delta}}(\sigma_{M_0}\overline{\sigma}_{M_0^c})]_R$ by Lemma 4.2. We then show that $\mathcal{E}(\sigma)$ is stable by intersection. Suppose $A, B \in \mathcal{E}(\sigma)$. Then clearly $A \cap B \supset [I_\delta(\sigma)]_R$ and using again Lemma 4.2,

$$A \supset [I_{\tilde{\delta}}(\sigma_A \overline{\sigma}_{A^c})]_R \supset [I_{\tilde{\delta}}(\sigma_{A\cap B}\overline{\sigma}_{(A\cap B)^c})]_R\,, \tag{4.17}$$

$$B \supset [I_{\tilde{\delta}}(\sigma_B \overline{\sigma}_{B^c})]_R \supset [I_{\tilde{\delta}}(\sigma_{A\cap B}\overline{\sigma}_{(A\cap B)^c})]_R\,, \tag{4.18}$$

which implies $A \cap B \in \mathcal{E}(\sigma)$. The following set is thus well defined, and is the candidate for describing the contours of the configuration σ:

$$I^*(\sigma) := \bigcap_{M \in \mathcal{E}(\sigma)} M\,. \tag{4.19}$$

By construction, $I^*(\sigma)$ is the smallest element of $\mathcal{E}(\sigma)$. A first important property of $I^*(\sigma)$ is the following, which will be essential to obtain the Peierls bound on the surface energy of contours.

Lemma 4.3. *There exists, in the $2R$-neighbourhood of each box $C^{(l)} \subset I^*(\sigma)$, a point $j \in I^*(\sigma)$ which is $\tilde{\delta}$-incorrect for the configuration $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$.*

Proof. Let $C^{(l)} \subset I^*(\sigma)$. First, suppose $I_\delta(\sigma) \cap [C^{(l)}]_{2R} \neq \emptyset$. Then each $j \in I_\delta(\sigma) \cap [C^{(l)}]_{2R}$ is δ-incorrect for σ, and hence $\tilde{\delta}$-incorrect for $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$, since $\tilde{\delta} < \delta$ and σ and $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$ coincide on $B_R(j)$.
Suppose there exists a box $C^{(l)}$ such that [3] $[I_\delta(\sigma)]_R \cap [C^{(l)}]_R = \emptyset$. If $I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) \cap [C^{(l)}]_{2R} = \emptyset$, i.e. $[I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_R \cap [C^{(l)}]_R = \emptyset$, then we define $I' := I^* \backslash C^{(l)}$ and show that $I' \in \mathcal{E}(\sigma)$, a contradiction with the definition of I^*. First, $I' \supset [I_\delta(\sigma)]_R$. Using Lemma 4.2, $I^* \supset [I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_R \supset [I_{\tilde{\delta}}(\sigma_{I'}\overline{\sigma}_{I'^c})]_R$. Since we have $[I_{\tilde{\delta}}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})]_R \cap [C^{(l)}]_R = \emptyset$, this implies $I' \supset [I_{\tilde{\delta}}(\sigma_{I'}\overline{\sigma}_{I'^c})]_R$, i.e. $I' \in \mathcal{E}(\sigma)$. $\square$

When studying restricted phases, we will need to re-sum over the set of configurations that have the same set of contours, that is to consider, for a fixed σ (we assume $I^*(\sigma) \neq \emptyset$),

$$\mathcal{A}(\sigma) := \left\{\sigma' : \sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)},\, I^*(\sigma') = I^*(\sigma)\right\}. \tag{4.20}$$

It is important to have an *explicit* characterisation of the set $\mathcal{A}(\sigma)$. Let $\Lambda^\#(\sigma)$ denote the set of points of $I^*(\sigma)^c$ that are $(\delta, \#)$-correct for σ. By Lemma 4.1 we have $d(\Lambda^+(\sigma), \Lambda^-(\sigma)) > l$, and we have the partition

$$\mathbf{Z}^d = I^*(\sigma) \cup \Lambda^+(\sigma) \cup \Lambda^-(\sigma)\,. \tag{4.21}$$

We now show that the set $\mathcal{A}(\sigma)$ can be characterised explicitly by

$$\mathcal{D}(\sigma) := \left\{\sigma' : \sigma'_{I^*(\sigma)} = \sigma_{I^*(\sigma)},\ \text{each } i \in [\Lambda^\#(\sigma)]_R \text{ is } (\delta, \#)\text{-correct for } \sigma'\right\}.$$

Proposition 4.1. *If $I^*(\sigma) \neq \emptyset$, then $\mathcal{A}(\sigma) = \mathcal{D}(\sigma)$.*

Proof. 1) Assume $\sigma' \in \mathcal{A}(\sigma)$. Then $I^* \equiv I^*(\sigma) = I^*(\sigma') \supset [I_\delta(\sigma')]_R$, so that each $i \in [I^{*c}]_R$ is δ-correct for σ'. Let A be a maximal connected component of $[I^{*c}]_R$. There exists $i \in A$ such that $i \in I^*$, since we assumed $I^* \neq \emptyset$. By Lemma 4.1, it suffices to show that i is $(\delta, +)$-correct for σ if and only if it is $(\delta, +)$-correct

[3] Here we use the fact that $A \cap [B]_{2R} = \emptyset$ if and only if $[A]_R \cap [B]_R = \emptyset$.

for σ'. Assume this is not the case, e.g. suppose i is $(\delta,+)$-correct for σ and $(\delta,-)$-correct for σ'. That is,

$$\sum_{j\neq i}|\omega_{ij}^{+}(-,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| = \sum_{j\in B_R^{\bullet}(i)\cap I^*}|w_{ij}^{+}(-,\sigma_j)| \leq \tilde{\delta}\,, \tag{4.22}$$

$$\sum_{j\neq i}|\omega_{ij}^{-}(+,(\sigma'_{I^*}\overline{\sigma}'_{I^{*c}})_j)| = \sum_{j\in B_R^{\bullet}(i)\cap I^*}|w_{ij}^{-}(+,\sigma_j)| \leq \tilde{\delta}\,. \tag{4.23}$$

Since $i \in I^*$ we have [4]

$$\sum_{j\in B_R^{\bullet}(i)\cap I^{*c}}|w_{ij}^{-}(+,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| \leq \sum_{j\in B_R^{\bullet}(i)\cap I^{*c}}|w_{ij}^{-}(+,+)| \leq 2(1-2^{-d})\,.$$

Therefore we get a contradiction, since,

$$\begin{aligned} 2 &= \sum_{j\neq i}|w_{ij}^{+}(-,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| + |w_{ij}^{-}(+,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| \\ &\leq 2\tilde{\delta} + 2\sum_{j\in B_R^{\bullet}(i)\cap I^{*c}}|w_{ij}^{-}(+,(\sigma_{I^*}\overline{\sigma}_{I^{*c}})_j)| \leq 2\tilde{\delta} + 2(1-2^{-d}) < 2\,, \end{aligned} \tag{4.24}$$

where we used the fact that $\tilde{\delta} < \delta < 2^{-d}$.
2) Suppose $\sigma' \in \mathcal{D}(\sigma)$. Since σ' coincides with σ on $I^*(\sigma)$ and all points of $[I^*(\sigma)^c]_R$ are δ-correct for σ', we have $I_\delta(\sigma') = I_\delta(\sigma)$. This gives $I^*(\sigma) \supset [I_\delta(\sigma)]_R = [I_\delta(\sigma')]_R$. Then, since $\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c} = \sigma'_{I^*(\sigma)}\overline{\sigma}'_{I^*(\sigma)^c}$, we have $I^*(\sigma) \supset [I_{\tilde{\delta}}(\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c})]_R = [I_{\tilde{\delta}}(\sigma'_{I^*(\sigma)}\overline{\sigma}'_{I^*(\sigma)^c})]_R$. This implies $I^*(\sigma) \in \mathcal{E}(\sigma')$, i.e. $I^*(\sigma') \subset I^*(\sigma)$. Assume $I^*(\sigma)\backslash I^*(\sigma') \neq \emptyset$. Using the fact that σ and σ' coincide on $I^*(\sigma)\backslash I^*(\sigma')$, we have $\sigma_{I^*(\sigma')}\overline{\sigma}_{I^*(\sigma')^c} = \sigma'_{I^*(\sigma')}\overline{\sigma}'_{I^*(\sigma')^c}$. This gives, like before, $I^*(\sigma') \supset [I_{\tilde{\delta}}(\sigma'_{I^*(\sigma')}\overline{\sigma}'_{I^*(\sigma')^c})]_R = [I_{\tilde{\delta}}(\sigma_{I^*(\sigma')}\overline{\sigma}_{I^*(\sigma')^c})]_R$. With $I^*(\sigma') \supset [I_\delta(\sigma')]_R = [I_\delta(\sigma)]_R$, this implies $I^*(\sigma') \in \mathcal{E}(\sigma)$, i.e. $I^*(\sigma') \supset I^*(\sigma)$. So $\sigma' \in \mathcal{A}(\sigma)$. □

In particular, Proposition 4.1 implies that $\sigma_{I^*(\sigma)}\overline{\sigma}_{I^*(\sigma)^c}$ is an element of $\mathcal{A}(\sigma)$.

Definition 4.2. *The connected components of $I^*(\sigma)$ form the support of the* contours *of the configuration σ, and are written* $\operatorname{supp}\Gamma_1,\ldots,\operatorname{supp}\Gamma_n$. *A* contour *is thus a couple $\Gamma = (\operatorname{supp}\Gamma, \sigma_\Gamma)$, where σ_Γ is the restriction of σ to Γ.*
A family of contours $\{\Gamma_1,\ldots,\Gamma_n\}$ is admissible *if there exists a configuration σ such that $\{\Gamma_1,\ldots,\Gamma_n\}$ are the contours of σ* [5].

[4] Here we use a property of the step function, but this can be done for any Kac potential whose function J has the symmetry $J(x) = J(y)$ when $\|x\| = \|y\|$.

[5] Note that the configuration σ is not unique, unlike in the usual situation treated in Pirogov-Sinai Theory.

The fact that the contours are defined on a coarse-grained scale will be crucial when dealing with their entropy, which we must control uniformly in γ. Notice that two (distinct) contours are at distance at least l from each other. We will usually denote $\operatorname{supp}\Gamma$ also by Γ. Contours should always be considered together with their type and labels, which we are about to define. The following topological property is needed for the definition of labels.

Lemma 4.4. *Fix $R \geq 1$. Let $B \subset \mathbf{Z}^d$ be R-connected and bounded. Then $\partial_R^+ A$ and $\partial_R^- A$ are R-connected, where A is any maximal R-connected component of $B^c = \mathbf{Z}^d \backslash B$.*

Proof. Let A be any maximal R-connected component of B^c. Then A^c is R-connected. Indeed, let $x, y \in A^c$, and consider a path $x_1 = x, x_2, \ldots, x_n = y$, $d(x_i, x_{i+1}) \leq R$. If $x_i \in A^c$ for all i there is nothing to show. So suppose there exists $1 \leq i_- < i_+ \leq n$ such that $\{x_1, \ldots, x_{i_- -1}, x_{i_-}\} \subset A^c$, $x_{i_-+1} \in A$, $x_{i_+-1} \in A$, $\{x_{i_+}, x_{i_++1}, \ldots, x_n\} \subset A^c$. Since A is maximal, we have $x_{i_-} \in B$, $x_{i_+} \in B$, and we can find a path from x_{i_-} to x_{i_+} entirely contained in B, i.e. in A^c.
We then show that $\partial_1^+ A$ is R-connected. Fix $\epsilon > 0$ and consider the sets

$$X = \left\{x \in \mathbb{R}^d : d(x, A) \leq \tfrac{R}{2} + \epsilon\right\}, \tag{4.25}$$
$$Y = \left\{y \in \mathbb{R}^d : d(y, A^c) \leq \tfrac{R}{2} + \epsilon\right\}. \tag{4.26}$$

Then X, Y are closed arc-wise connected subsets of $\mathbb{R}^d$, and $X \cup Y = \mathbb{R}^d$. By a Theorem of Kuratowski, $X \cap Y$ is arc-wise connected [6]. Let $\epsilon' > 0$ and consider $x, y \in \partial_1^+ A$, together with $\tilde{x}, \tilde{y} \in X \cap Y$ such that $d(x, \tilde{x}) < \frac{1}{2}$, $d(y, \tilde{y}) < \frac{1}{2}$. Then consider any sequence $\tilde{x}_1 = \tilde{x}, \ldots, \tilde{x}_n = \tilde{y}$, $\tilde{x}_i \in X \cap Y$, $d(\tilde{x}_i, \tilde{x}_{i+1}) \leq \epsilon'$. For each i we have $d(\tilde{x}_i, A) \leq \frac{R}{2} + \epsilon$, $d(\tilde{x}_i, A^c) \leq \frac{R}{2} + \epsilon$. This implies that each box $B_{\frac{R}{2}+\epsilon}(\tilde{x}_i)$ contains at least one element $x_i' \in \partial_1^+ A$, i.e. $d(\tilde{x}_i, x_i') \leq \frac{R}{2} + \epsilon$. We have

$$d(x_i', x_{i+1}') \leq d(x_i', \tilde{x}_i) + d(\tilde{x}_i, \tilde{x}_{i+1}) + d(\tilde{x}_{i+1}, x_{i+1}') \leq R + 2\epsilon + \epsilon'. \tag{4.27}$$

If $2\epsilon + \epsilon' < \frac{1}{2}$, this shows that $\partial_1^+ A$ is R-connected, which implies that $\partial_R^+ A$ is R-connected. The same proof holds when $\partial_R^+ A$ is replaced by $\partial_R^- A$. □

Let Γ be a contour of σ, A a maximal R-connected component of $(\operatorname{supp}\Gamma)^c$. Let $i \in \partial_R^- A$. By definition, i is $(\delta, \#)$-correct for σ for some $\# \in \{\pm 1\}$. By Lemmas 4.4 and 4.1, each $i' \in \partial_R^- A$ is $(\delta, \#)$-correct for σ for the *same* value $\#$. We call $\#$ the **label** of the component A. There exists a unique unbounded component of Γ^c. The label of this component is called the **type** of the contour Γ. Let Γ be of type $+$ (resp. $-$). The union of all components of Γ^c with label $-$ (resp. $+$) is called the **interior** of Γ, and is denoted $\operatorname{int}\Gamma$. Notice that there is only one type

[6] This property of $\mathbb{R}^d$ is called *unicoherence*. See [Ku], vol. 2, Theorem 9 of Chapter 57.I, and Theorem 2 of Chapter 57.II.

of interior. We define $V(\Gamma) := |\mathrm{int}\Gamma|$. The union of the remaining components is called the exterior of Γ, and is denoted by $\mathrm{ext}\Gamma$. A contour is external if it is not contained in the interior of another contour.

Let Γ be a contour of some configuration σ. Assume Γ is of type $+$. Consider the configuration $\sigma[\Gamma]$, which coincides with σ_Γ on the support of Γ, and which equals $+1$ on $\mathrm{ext}\Gamma$, -1 on $\mathrm{int}\Gamma$. Using Proposition 4.1, it is easy to see that $\sigma[\Gamma]$ has a single contour, which is exactly Γ. This can be generalised to a family of external contours of the same type, as in the second part of the following lemma.

Lemma 4.5. *External contours have the following properties:*
1) External contours of an admissible family have the same type.
2) Let $\{\Gamma_1, \dots, \Gamma_n\}$ be a family of external contours, all of the same type. Then $\{\Gamma_1, \dots, \Gamma_n\}$ is admissible if and only if $d(\Gamma_i, \Gamma_j) > l$ for all $i \neq j$.

Proof. The first statement follows easily from Lemma 4.4. For the second, we can assume that the contours are of type $+$. If $\{\Gamma_1, \dots, \Gamma_n\}$ is admissible, then by construction the Γ_i are at distance at least l. Then, assume $d(\Gamma_i, \Gamma_j) > l$ for all $i \neq j$. Consider the configuration $\sigma[\Gamma_1, \dots, \Gamma_n]$, which coincides with σ_{Γ_i} on the support of Γ_i, which equals $+1$ on $\bigcap_i \mathrm{ext}\Gamma_i$ and -1 on $\bigcup_i \mathrm{int}\Gamma_i$. Then the contours of $\sigma[\Gamma_1, \dots, \Gamma_n]$ are given by $\{\Gamma_1, \dots, \Gamma_n\}$. □

4.2 Re-formulation of the Hamiltonian

Consider a finite volume $\Lambda \in \mathcal{C}^{(l)}$ with the pure $+$-boundary condition $+_{\Lambda^c} \in \Omega_{\Lambda^c}$. Let $\sigma_\Lambda \in \Omega_\Lambda$. We set $\sigma := \sigma_\Lambda +_{\Lambda^c}$. The hamiltonian with boundary condition $+_{\Lambda^c}$ is defined by

$$H_\Lambda(\sigma) = H_\Lambda(\sigma_\Lambda +_{\Lambda^c}) = \sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset \\ i\neq j}} \phi_{ij}(\sigma_i, \sigma_j) + \sum_{i\in\Lambda} u(\sigma_i)\,, \tag{4.28}$$

where $u(\sigma_i) = -h\sigma_i$, $h \in \mathbb{R}$. Since we work in a finite volume, we will from now on identify $I^*(\sigma)$ with $I^*(\sigma)\cap\Lambda$ and $\Lambda^\pm(\sigma)$ with $\Lambda^\pm(\sigma)\cap\Lambda$. The following lemma shows how the hamiltonian can be written in such a way that spins in correct regions interact via the functions $w^\#_{ij}$ and are subject to an effective external field $U^\#$.

Lemma 4.6. *Define the potential $U^\#(\sigma_i) := u(\sigma_i) + \sum_{j:j\neq i} \phi_{ij}(\sigma_i, \#)$. Suppose σ_Λ is such that $I^*(\sigma) \cap \partial_R^- \Lambda = \emptyset$. Then*

$$H_\Lambda(\sigma) = H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) + \sum_{\#} \Big(\sum_{\substack{\{i,j\}\cap\Lambda^\#\neq\emptyset \\ i\neq j}} w^\#_{ij}(\sigma_i, \sigma_j) + \sum_{i\in\Lambda^\#} U^\#(\sigma_i) \Big)\,. \tag{4.29}$$

Proof. The proof is a simple rearrangement of the terms. Consider a pair $\{i,j\}$ appearing in $H_\Lambda(\sigma)$. Since $d(\Lambda^+,\Lambda^-) > R$ we have a certain number of cases to consider: 1) $\{i,j\} \subset \Lambda^+$. In this case, write

$$\phi_{ij}(\sigma_i,\sigma_j) = w^+_{ij}(\sigma_i,\sigma_j) + \phi_{ij}(\sigma_i,+) + \phi_{ij}(+,\sigma_j)\,. \tag{4.30}$$

The second term contributes to $U^+(\sigma_i)$, the third to $U^+(\sigma_j)$. 2) $i \in \Lambda^+$, $j \in I^*$. In this case the third term contributes to $H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}})$. 3) $i \in \Lambda^+$, $j \in \Lambda^c$; in this case, $\phi_{ij}(+,\sigma_j) = 0$. The other cases are similar. Notice that the case $i \in \Lambda^-$, $j \in \Lambda^c$ never occurs since points of $\partial^-_R\Lambda$ can only be $(\delta,+)$-correct. □

4.3 The Peierls Condition

We take a closer look at the term H_{I^*}. Remember that contours are maximal R-connected components of I^*. For each contour Γ, $\sigma[\Gamma]$ and $\sigma_{I^*}\overline{\sigma}_{I^{*c}}$ coincide on $[I^*]_R$. Since $d(\Gamma,\Gamma') > l$, we can decompose

$$H_{I^*}(\sigma_{I^*}\overline{\sigma}_{I^{*c}}) = \sum_\Gamma H_\Gamma(\sigma[\Gamma]) \tag{4.31}$$

$$= \sum_\Gamma \Big(\|\Gamma\| + \sum_{i\in\Gamma} u(\sigma[\Gamma]_i)\Big)\,, \tag{4.32}$$

where the sum is over contours of the configuration σ (contained in Λ), and where the surface energy is defined by

$$\|\Gamma\| := \sum_{\substack{\{i,j\}\cap\Gamma\neq\emptyset\\ i\neq j}} \phi_{ij}(\sigma[\Gamma]_i,\sigma[\Gamma]_j)\,. \tag{4.33}$$

The central result of this section is the following.

Proposition 4.2. *Let $0 < \delta < 2^{-d}$, $0 < \tilde\delta < \delta$ be fixed, small. The surface energy satisfies the* Peierls condition*, i.e. there exists $\rho = \rho(\tilde\delta,\nu) > 0$ such that for all contour Γ,*

$$\|\Gamma\| \geq \rho|\Gamma|\,. \tag{4.34}$$

The constant ρ is independent of γ and is called the Peierls constant*.*

Notice that $|\Gamma|$ denotes the total number of lattice sites contained in the support of Γ [7]. To show Proposition 4.2, we need two lemmas. The first is purely geometric.

Lemma 4.7. *For any finite set $A \subset \mathbf{Z}^d$ and for all $R_0 \in \mathbb{N}$, there exists $A_0 \subset A$, called an R_0-*approximant of *A, such that $A \subset [A_0]_{R_0}$, and $d(x,y) > R_0$ for all $x,y \in A_0$, $x \neq y$.*

[7] In the litterature, $|\Gamma|$ often denotes the number of *blocks* contained in Γ.

The second lemma is a property of the Kac potential. (In [BZ2], this property was called "continuity" for obvious reasons.)

Lemma 4.8. *Let $\sigma \in \Omega$, $i \in \mathbf{Z}^d$, $\# \in \{\pm\}$. Define*

$$V_\sigma(i;\#) := \sum_{j:j\neq i} \phi_{ij}(\#,\sigma_j)\,. \tag{4.35}$$

Then there exists $c_2 > 0$ such that for all x, y, $d(x,y) \leq R$,

$$|V_\sigma(x;\#) - V_\sigma(y;\#)| \leq c_2 \frac{d(x,y)}{R}\,. \tag{4.36}$$

Proof. The difference $V_\sigma(x;\#) - V_\sigma(y;\#)$ can be expressed as follows:

$$\sum_{\substack{j\in B_R(x)\\ j\notin B_R(y)}} \phi_{xj}(\#,\sigma_j) + \sum_{\substack{j\in B_R(x)\cap B_R(y)\\ j\neq x,\, j\neq y}} \big(\phi_{xj}(\#,\sigma_j) - \phi_{yj}(\#,\sigma_j)\big) - \sum_{\substack{j\in B_R(y)\\ j\notin B_R(x)}} \phi_{yj}(\#,\sigma_j)$$

The first and last sum can be estimated as follows:

$$\sum_{\substack{j\in B_R(x)\\ j\notin B_R(y)}} \phi_{xj}(\#,\sigma_j) \leq \big(|B_R(x)| - |B_R(x)\cap B_R(y)|\big) \sup \phi_{ij} \tag{4.37}$$

$$\leq d c_\gamma \big(\sup_t J(t)\big) \Big(\frac{2R+1}{R}\Big)^{d-1} \frac{d(x,y)}{R}\,. \tag{4.38}$$

Since we are considering the step function (3.14), $\sup_t J(t) = 2^{-d}$, and the middle sum vanishes [8], which finishes the proof. □

Proof of Proposition 4.2: By Lemma 4.3 there exists in the $2R$-neighbourhood of each $C^{(l)} \subset \Gamma$ a point $j \in \Gamma$ that is $\tilde{\delta}$-incorrect for $\sigma[\Gamma]$. Let A be the set of all such points. We have $\Gamma \subset [A]_{l+2R}$. Let A_0 be any $4R$-approximant of A. We have $A \subset [A_0]_{4R}$, i.e. $\Gamma \subset [A_0]_{l+6R}$. Each $j \in A_0$ is $\tilde{\delta}$-incorrect for $\sigma[\Gamma]$ i.e. satisfies

$$\sum_{k:k\neq j} |w^{\pm}_{jk}(\mp,\sigma[\Gamma]_k)| > \tilde{\delta}\,. \tag{4.40}$$

[8] If J is an arbitrary K-Lipshitz function:

$$\sum_{j\in B_R(x)\cap B_R(y)} \big|\phi_{xj}(\#,\sigma_j) - \phi_{yj}(\#,\sigma_j)\big| \leq K c_\gamma \gamma^d \sum_{j\in B_R(x)\cap B_R(y)} d(\gamma x,\gamma y)$$

$$\leq K c_\gamma \gamma^d |B_R(x)| \frac{d(x,y)}{R}\,. \tag{4.39}$$

Since $|w^{\pm}_{jk}(\mp,\sigma[\Gamma]_k)| = 2\phi_{jk}(\pm,\sigma[\Gamma]_k)$,

$$V_{\sigma[\Gamma]}(j;\pm) = \sum_{k:k\neq j} \phi_{jk}(\pm,\sigma[\Gamma]_k) > \frac{\tilde{\delta}}{2}\,. \tag{4.41}$$

We bound the surface energy from below as follows:

$$\begin{aligned}
\|\Gamma\| &\geq \frac{1}{2}\sum_{j\in A_0}\sum_{k\in B_R(j)\cap\Gamma}\sum_{l:l\neq k}\phi_{kl}(\sigma[\Gamma]_k,\sigma[\Gamma]_l)\\
&= \frac{1}{2}\sum_{j\in A_0}\sum_{k\in B_R(j)\cap\Gamma} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) \geq \frac{1}{2}\sum_{j\in A_0}\sum_{k\in B_R(j)\cap C_j^{(l)}} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k)\\
&\geq \frac{1}{2}\sum_{j\in A_0}\sum_{\substack{k\in B_R(j)\cap C_j^{(l)}\\ d(k,j)\leq\frac{\tilde{\delta}}{4c_2}R}} V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k)\,,
\end{aligned}$$

where c_2 was defined in Lemma 4.8, and we assumed $\tilde{\delta}$ is small enough. Using (4.36) for each k of the sum,

$$\begin{aligned}
V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) &= V_{\sigma[\Gamma]}(j;\sigma[\Gamma]_k) + \big(V_{\sigma[\Gamma]}(k;\sigma[\Gamma]_k) - V_{\sigma[\Gamma]}(j;\sigma[\Gamma]_k)\big) &(4.42)\\
&\geq \frac{\tilde{\delta}}{2} - c_2\frac{d(k,j)}{R} \geq \frac{\tilde{\delta}}{2} - c_2\frac{\tilde{\delta}}{4c_2} = \frac{\tilde{\delta}}{4}\,. &(4.43)
\end{aligned}$$

We used the important fact that the correctness of a point j does not depend on the value taken by the spin σ_j. This gives the lower bound

$$\|\Gamma\| \geq \frac{1}{2}|A_0|\frac{1}{2^d}|B_{\frac{\tilde{\delta}}{4c_2}R}(0)|\frac{\tilde{\delta}}{4} \geq \frac{\tilde{\delta}}{2^{d+3}}|B_{\frac{\tilde{\delta}}{4c_2}R}(0)||B_{l+6R}(0)|^{-1}|\Gamma| \geq \rho|\Gamma|\,.$$

□

4.4 Isoperimetric Constants

We discussed in the Introduction of the importance of the isoperimetric ratios $\frac{V(\Gamma)}{\|\Gamma\|}$. These will play a fundamental role in the proof of non-analyticity of the pressure at $h = 0$. In the present section, we discuss more precisely a few isoperimetric problems and their associated constants.

To start with, we give a purely geometrical (model-independent) result. It is essentially a consequence of the standard isoperimetric inequality on $\mathbf{Z}^d$.

Lemma 4.9. *Let $B \in \mathcal{C}^{(l)}$, and let A be the union of all finite maximal R-connected component of B^c. Then*

$$|B| \geq |\partial_l^+ A| \geq l|A|^{\frac{d-1}{d}}. \tag{4.44}$$

Proof. Consider the edge boundary $\delta^+ A := \{e = \langle i,j\rangle : i \in A, j \in A^c\}$, where $\langle i,j\rangle$ means that i,j are nearest neighbours. Decompose $\delta^+ A = E_1 \cup \cdots \cup E_d$, where E_α is the set of edges of $\delta^+ A$ that are parallel to the coordinate axis α. Suppose $e = \langle i,j\rangle$, $i \in A$, $j \in A^c$. Since A is maximal, $C_j^{(l)} \subset B$. Moreover,

$$T_e := \Big\{j, j+(j-i), j+2(j-i), \dots, j+(\frac{l}{2}-1)(j-i)\Big\} \subset B. \tag{4.45}$$

For all $e, e' \in E_\alpha$, $T_e \cap T_{e'} = \emptyset$. So for all α,

$$|\partial_l^+ A| \geq \Big|\bigcup_{e \in E_\alpha} T_e\Big| = \sum_{e \in T_\alpha} |T_e| = \frac{l}{2}|E_\alpha|. \tag{4.46}$$

Considering the inequality $|\delta^+ A| \leq d \max_\alpha |E_\alpha|$ and the standard isoperimetric inequality $|\delta^+ A| \geq 2d|A|^{\frac{d-1}{d}}$ finishes the proof. □

By Proposition 4.2 and Lemma 4.9, we have

$$\frac{V(\Gamma)}{\|\Gamma\|} \leq \frac{V(\Gamma)}{\rho|\Gamma|} \leq \frac{1}{\rho l} V(\Gamma)^{\frac{1}{d}}. \tag{4.47}$$

Since the Peierls constant ρ is not optimal, there certainly doesn't exist contours that saturate inequality (4.47). That is, $(\rho l)^{-1}$ can be considered as a "bad" isoperimetric constant.

The Assumption of Isakov. In his second paper [9], Isakov introduced the following numbers ($N \in \mathbb{N}$):

$$\tau(N) := \sup\Big\{\frac{V(\Gamma)}{\|\Gamma\|} : \forall \Gamma,\, V(\Gamma) \leq N\Big\}. \tag{4.48}$$

By (4.47) we have an upper bound $\tau(N) \leq (\rho l)^{-1} N^{\frac{1}{d}}$, and a lower bound can be obtained easily, by choosing a cubical contour. We get

$$\alpha_- N^{\frac{1}{d}} \leq \tau(N) \leq \alpha_+ N^{\frac{1}{d}}, \tag{4.49}$$

[9] In [Isakov2], Isakov was considering a two phase model, different from the Kac Model, where the phases are not necessarily related by symmetry. He studied the phase diagram with the help of *two* functions $\tau_1(N), \tau_2(N)$, one for each phase. Here we are in the case where only one function is necessary, because of symmetry.

with $\alpha_-, \alpha_+ > 0$. Isakov constructed the phase diagram of his model with the help of the numbers $\tau(N)$, but he showed non-analyticity at the transition point only under the following assumption, which requires the exact asymptotic behaviour of $\tau(N)$ to be known.

Assumption ([Isakov2]): *The following limit exists:*

$$\lim_{N\to\infty} N^{-\frac{1}{d}}\tau(N) = \alpha\,. \tag{4.50}$$

For a given model, in particular in the case we are considering, the existence of the limit (4.50) seems very hard, if not impossible, to check. The difficulty lies in the fact that the surface energy $\|\Gamma\|$ depends on the way in which the contours were defined and on the details of the underlying interactions of the hamiltonian.

Before showing how this problem can be avoided, let us explicit the implications of (4.50). By definition of $\tau(N)$, there exists for all N a contour Γ_N such that $V(\Gamma_N) \leq N$ and

$$\tau(N) = \tau(V(\Gamma_N)) = \frac{V(\Gamma_N)}{\|\Gamma_N\|}\,. \tag{4.51}$$

Using the bounds (4.49), we get

$$\Big(\frac{\alpha_-}{\alpha_+}\Big)^d N \leq V(\Gamma_N) \leq N\,. \tag{4.52}$$

By (4.50), this implies $\lim_N V(\Gamma_N)^{-\frac{1}{d}}\tau(V(\Gamma_N)) = \alpha$. Using (4.51), this shows that the assumption of Isakov implies the existence, for all $\epsilon > 0$, of a sequence $(\Gamma_N)_{N\geq 1}$, $V(\Gamma_N) \nearrow +\infty$, such that for large enough N,

$$(1-\epsilon)\alpha V(\Gamma_N)^{\frac{1}{d}} \leq \frac{V(\Gamma_N)}{\|\Gamma_N\|} \leq (1+\epsilon)\alpha V(\Gamma_N)^{\frac{1}{d}}\,. \tag{4.53}$$

We call each contour Γ_N a maximising contour, in the sense that it saturates some isoperimetric inequality. The fact that the elements of the sequence $(\Gamma_N)_{N\geq 1}$ satisfy $V(\Gamma_N) \nearrow +\infty$ is crucial for obtaining non-analyticity: it implies that the maximisers can be of arbitrary large size [10]. In fact, the assumption of Isakov could be formulated in the following way, which is slightly weaker but gives the same results. Let $C(N)$ be the bounded increasing sequence of isoperimetric constants defined by

$$C(N) := \inf\Big\{\kappa > 0 : \frac{V(\Gamma)}{\|\Gamma\|} \leq \kappa V(\Gamma)^{\frac{1}{d}}\ \forall\Gamma\,, V(\Gamma) \leq N\Big\}\,. \tag{4.54}$$

[10] This will imply the existence of an *unbounded* sequence $k_1, k_2, \ldots$ such that the k_i-th derivative of the pressure at $h = 0$ behaves like $k_i!^{\frac{d}{d-1}}$.

Let $C(\infty) := \lim_{N\to\infty} C(N)$.

Assumption (variant): *There exists for all $\epsilon > 0$ a sequence of maximising contours $(\Gamma_N)_{N\geq 1}$ such that $V(\Gamma_N) \nearrow +\infty$ and for large enough N,*

$$(1-\epsilon)C(\infty)V(\Gamma_N)^{\frac{1}{d}} \leq \frac{V(\Gamma_N)}{\|\Gamma_N\|} \leq (1+\epsilon)C(\infty)V(\Gamma_N)^{\frac{1}{d}}\,. \tag{4.55}$$

Redefining the Isoperimetric Constants. Our extension of the results of Isakov is possible after considering the following isoperimetric constants (compare with (4.54)):

$$K(N) := \inf\Big\{\kappa > 0 : \frac{V(\Gamma)}{\|\Gamma\|} \leq \kappa V(\Gamma)^{\frac{1}{d}}\ \forall\Gamma\,, V(\Gamma)\geq N\Big\}\,. \tag{4.56}$$

The advantage of the constants $K(N)$, as can be seen in the following lemma, is that they are defined in such a way that the assumption of Isakov (or, rather, its variant) is always satisfied; the precise structure of the maximisers need not be considered in details.

Lemma 4.10. *The sequence $K(N)$ is decreasing and there exists positive constants c_-, c_+ such that*

$$c_-\gamma \leq \inf_N K(N) \leq \sup_N K(N) \leq c_+\gamma\,. \tag{4.57}$$

As a consequence, the following limit exists

$$K(\infty) := \lim_{N\to\infty} K(N)\,. \tag{4.58}$$

Moreover, there exists for all $\epsilon > 0$ a sequence $(\Gamma_N)_{N\geq 1}$, $V(\Gamma_N) \nearrow +\infty$, such that for N large enough,

$$(1-\epsilon)K(\infty)V(\Gamma_N)^{\frac{1}{d}} \leq \frac{V(\Gamma_N)}{\|\Gamma\|} \leq (1+\epsilon)K(\infty)V(\Gamma_N)^{\frac{1}{d}}\,. \tag{4.59}$$

Proof. $K(N)$ is decreasing by definition. For the upper bound, use the Peierls condition and Lemma 4.9: for all Γ,

$$\frac{V(\Gamma)^{\frac{d-1}{d}}}{\|\Gamma\|} \leq \frac{V(\Gamma)^{\frac{d-1}{d}}}{\rho|\Gamma|} \leq \frac{1}{\rho l} = \frac{1}{\rho\nu}\gamma \equiv c_+\gamma\,. \tag{4.60}$$

For the lower bound, we explicitly construct a large contour of cubic shape. Fix N and take $M \in \mathbb{N}$ large, so that $\Lambda_M = [-M;+M]^d \cap \mathbf{Z}^d$, $\Lambda_M \in \mathcal{C}^{(l)}$,

$|\Lambda_M| \geq 2N$. Consider the configuration σ defined by $\sigma_i = -1$ if $i \in \Lambda_M$, $\sigma_i = +1$ if $i \in \Lambda_M^c$. Clearly, $I^*(\sigma)$ contains a single contour Γ_M (of type $+$). Using (4.6), $\|\Gamma_M\| \leq |\Gamma_M| \leq 2l|\partial_1^+\Lambda_M| = 2\nu R|\partial_1^+\Lambda_M|$. Taking M large enough guarantees $|\Lambda_M| \geq V(\Gamma_M) \geq \frac{1}{2}|\Lambda_M|$. This gives, since $|\partial_1^+\Lambda_M| = 2d|\Lambda_M|^{\frac{d-1}{d}}$,

$$\frac{V(\Gamma_M)}{\|\Gamma_M\|} \geq \frac{1}{2}\frac{1}{2\nu R}\frac{|\Lambda_M|}{|\partial_1^+\Lambda_M|} \geq \frac{\gamma}{8d\nu}V(\Gamma_M)^{\frac{1}{d}} \equiv c_-\gamma V(\Gamma_M)^{\frac{1}{d}}\,. \tag{4.61}$$

The existence of the sequence $(\Gamma_N)_{N\geq 1}$ follows from the definition of $K(N)$ and from the existence of the limit $K(\infty)$. □

Chapter 5

Restricted Phases

Restricted phases are the analog of the ground states of Pirogov-Sinai Theory. The difference with ground states is that they have a non-trivial pressure. We use them in order to obtain results that are uniform in the range of interaction. The study of restricted phases we present was invented by Bovier and Zahradník in [BZ2]. At a few places our development differs slightly from theirs, so we expose all the details.

In our analysis of the phase diagram, restricted phases will intervene when we re-sum over all the configurations that have the same set of contours. Since the set of configurations having the same set of contours was completely characterised in Proposition 4.1, we are naturally led to consider systems living in a volume Λ with a boundary condition η_{Λ^c}, with the constraint that each point $i \in [\Lambda]_R$ must be δ-correct. Our aim is to obtain a polymer representation for the partition function of such systems, and to show that the associated pressure behaves analytically at $h = 0$. As will be seen, the presence of the constraint will allow to treat the system in a way very similar to a high temperature expansion.

A source of complication will be that the definition of polymers, as well as their weights, will depend on the boundary conditions specified outside Λ. Typically, the Λ we want to consider is the volume between a given set of contours and the boundary of a box. That is, the boundary condition is specified partly by the spins on the contours and partly by the boundary condition outside the box. To have an idea of the objects that will be constructed in this chapter, see Figure 5.1.

We will only treat the case $+$, the case $-$ being similar by symmetry. Fix $0 < \tilde{\delta} < \delta < 2^{-d}$. Consider any finite set $\Lambda \in \mathcal{C}^{(l)}$. First of all, we must consider boundary conditions of the following type:

Definition 5.1. *A boundary condition $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$ is* $+$-admissible *if each $i \in [\Lambda]_R$ is $(\tilde{\delta}, +)$-correct for the configuration $+_\Lambda \eta_{\Lambda^c}$.*

More intuitively, a +-admissible boundary condition means that when looked from any point i inside of Λ, there is a majority of spins $+1$ on the boundary. In our case (i.e. with the step function), this can be formulated as follows: for each $i \in [\Lambda]_R$,

$$|B_R^\bullet(i) \cap B| \le \tfrac{\tilde{\delta}}{2}|B_R(i)|\,, \tag{5.1}$$

where the set B is defined by

$$B = B(\eta_{\Lambda^c}) := \{i \in \Lambda^c : (\eta_{\Lambda^c})_i = -1\}\,. \tag{5.2}$$

In this sense, +-admissible boundary conditions are "good"; there is hope in being able to control the +-phase in the volume Λ. Notice that the boundary condition specified by a contour on its interior is always admissible. This is the reason why the parameter $\tilde{\delta}$ was introduced in their definition.

We define the function that allows to realise the constraint obtained after Proposition (4.1): consider a +-admissible boundary condition $\eta_{\Lambda^c} \in \Omega_{\Lambda^c}$. Let $i \in [\Lambda]_R$, $\sigma_\Lambda \in \Omega_\Lambda$, and define

$$1_i(\sigma_\Lambda) := \begin{cases} 1 & \text{if } i \text{ is } (\delta,+)\text{-correct for } \sigma_\Lambda\eta_{\Lambda^c} \\ 0 & \text{otherwise .} \end{cases} \tag{5.3}$$

Then define

$$1(\sigma_\Lambda) := \prod_{i \in [\Lambda]_R} 1_i(\sigma_\Lambda)\,. \tag{5.4}$$

Notice that $1(+_\Lambda) = 1$ since η_{Λ^c} is +-admissible. Remember the functions $w_{ij}^{\#}$ defined in (4.7). The hamiltonian we use for the restricted system is the one obtained after the re-formulation of Lemma 4.6 in a region of +-correct points. Set $\sigma := \sigma_\Lambda\eta_{\Lambda^c}$. The restricted partition function with boundary condition η_{Λ^c} is

$$Z_r^{\ +}(\Lambda;\eta_{\Lambda^c}) := \sum_{\sigma_\Lambda \in \Omega_\Lambda} 1(\sigma_\Lambda) \exp\Big(-\beta \sum_{\substack{\{i,j\}\cap\Lambda \neq \emptyset \\ i \neq j}} w_{ij}^{+}(\sigma_i,\sigma_j) - \beta \sum_{i \in \Lambda} U^{+}(\sigma_i)\Big)\,.$$

The aim of this chapter is to show that $Z_r^{\ +}$ can be put in the form $Z_r^{\ +} = e^{\beta h|\Lambda|}\mathcal{Z}_r$, where $\mathcal{Z}_r$ is the partition function of a polymer model, having a normally convergent cluster expansion in the domain

$$H_+ = \big\{h \in \mathbb{C} : \operatorname{Re} h > -\tfrac{1}{8}\big\}\,. \tag{5.5}$$

As will be seen, the reason for $\log {Z_r}^+$ to behave analytically at $h = 0$ is that the presence of contours is suppressed by $1(\sigma_\Lambda)$, and that on each spin $\sigma_i = -1$ acts an effective magnetic field

$$U^+(-1) = h + \sum_{j:j\neq i} \phi_{ij} = 1 + h\,, \tag{5.6}$$

which is close to 1 when h is in a neighbourhood of $h = 0$.

Conventions. We will often use the norm

$$\|f\|_D := \sup_{z\in D} |f(z)|\,.$$

When G is a graph we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges.

5.1 Representation with Polymers

The influence of a boundary condition can always be interpreted as a magnetic field acting on sites near the boundary. We thus rearrange the terms of the hamiltonian as follows:

$$\begin{aligned}\sum_{\substack{\{i,j\}\cap\Lambda\neq\emptyset\\ i\neq j}} & w^+_{ij}(\sigma_i,\sigma_j) + \sum_{i\in\Lambda} U^+(\sigma_i)\\ &= \sum_{\substack{\{i,j\}\subset\Lambda\\ i\neq j}} w^+_{ij}(\sigma_i,\sigma_j) + \sum_{i\in\Lambda}\Big(U^+(\sigma_i) + \sum_{j\in\Lambda^c} w^+_{ij}(\sigma_i,(\eta_{\Lambda^c})_j)\Big)\,. \end{aligned}\tag{5.7}$$

By defining a new effective non-homogeneous magnetic field

$$\mu^+_i(\sigma_i) := U^+(\sigma_i) + h + \sum_{j\in\Lambda^c} w^+_{ij}(\sigma_i,(\eta_{\Lambda^c})_j)\,, \tag{5.8}$$

we can extract a volume term from ${Z_r}^+$ and write ${Z_r}^+ = e^{\beta h|\Lambda|}\mathcal{Z}_r$, where

$$\mathcal{Z}_r := \sum_{\sigma_\Lambda\in\Omega_\Lambda} 1(\sigma_\Lambda)\exp\Big(-\beta\sum_{\substack{\{i,j\}\subset\Lambda\\ i\neq j}} w^+_{ij}(\sigma_i,\sigma_j) - \beta\sum_{i\in\Lambda}\mu^+_i(\sigma_i)\Big)\,. \tag{5.9}$$

Notice that the field $\mu^+_i(\sigma_i)$ becomes independent of η_{Λ^c} when $d(i,\Lambda^c) > R$. Since $w^+_{ij}(\sigma_i,\sigma_j) = 0$ if $\sigma_i = +1$ or $\sigma_j = +1$ and $\mu^+_i(+1) = 0$, we need only consider points i with $\sigma_i = -1$, which will be identified with the vertices of a graph. Each vertex of this graph will then get a factor $e^{-\beta\mu_i^+(-1)}$. When $h \in H_+$, we can use the fact that η_{Λ^c} is +-admissible and that $\tilde{\delta} < 2^{-d}$:

$$\mathrm{Re}\,\mu_i^+(-1) = 1 + 2\mathrm{Re}\,h + \sum_{j\in\Lambda^c} w_{ij}^+(-,(\eta_{\Lambda^c})_j) \geq 1 - 2\tfrac{1}{8} - \tilde{\delta} > \tfrac{1}{2}\,. \tag{5.10}$$

The formulation of $\mathcal{Z}_r$ in terms of polymers will be a three step procedure. We first express $\mathcal{Z}_r$ as a sum over graphs, satisfying a certain constraint inherited from $1(\sigma_\Lambda)$. Then, we associate to each graph a spanning tree and re-sum over all graphs having the same spanning tree. We will see that the weights of the trees obtained have good decreasing properties. Finally, the constraint is expanded, yielding sets on which the constraint is *violated.* These sets are linked with trees. After a second partial re-summation, this yields a sum over polymers, which are nothing but particular graphs with vertices living on $\mathbf{Z}^d$ and whose edges are of length at most R.

A sum over graphs. Let $\mathcal{G}_\Lambda$ be the family of simple non-oriented graphs $G = (V, E)$ where $V \subset \Lambda$, each edge $e = \{i, j\} \in E$ has $d(i, j) \leq R$. For $e = \{i, j\}$, set $w_e^+ := w_{ij}^+(-,-)$. Notice that $\omega_e^+ = -2\phi_{ij} \leq 0$. Define also $\mu_i^+ := \mu_i^+(-1)$. Expanding the product over edges leads to the following expression

$$\mathcal{Z}_r = \sum_{G\in\mathcal{G}_\Lambda} 1(V(G)) \prod_{e\in E(G)} (e^{-\beta w_e^+} - 1) \prod_{i\in V(G)} e^{-\beta\mu_i^+}\,, \tag{5.11}$$

where $1(V) := 1(\sigma_\Lambda(V))$, and $\sigma_\Lambda(V) \in \Omega_\Lambda$ is defined by $\sigma_\Lambda(V)_i = -1$ if $i \in V$, $+1$ otherwise. With this formulation in terms of graphs, the constraint $1(V(G)) = 1$ is satisfied if and only if

$$\sum_{\substack{e=\{i,j\}\\ j\in V(G)\cup B}} |w_e^+| \leq \delta \qquad \forall i \in [\Lambda]_R\,. \tag{5.12}$$

Moreover, the fact that the boundary condition η_{Λ^c} is +-admissible reduces to

$$\sum_{\substack{e=\{i,j\}\\ j\in B}} |w_e^+| \leq \tilde{\delta}\,. \tag{5.13}$$

A sum over trees. Suppose we are given an algorithm that assigns to each connected graph G_0 a deterministic spanning tree $T(G_0)$, in a translation invariant way. That is if G_0' is obtained from G_0 by translation then $T(G_0')$ is obtained from $T(G_0)$ by the same translation. To be precise, we consider the Penrose algorithm considered in Chapter 3 of [Pf] [1]. We apply the Penrose algorithm to

[1] The Penrose algorithm requires the choice of an origin among the vertices of the graph. We choose this origin as the smallest vertex of the graph with respect to the lexicographical order.

each component of each graph G appearing in the partition function (5.11). Let $\mathcal{T}_\Lambda \subset \mathcal{G}_\Lambda$ denote the set of all forests. We have

$$\mathcal{Z}_r = \sum_{T \in \mathcal{T}_\Lambda} 1(V(T)) \prod_{\mathtt{t} \in T} \omega^+(\mathtt{t}) \,, \tag{5.14}$$

where the product is over the trees of T, and the weight of each tree is defined by

$$\omega^+(\mathtt{t}) := \sum_{\substack{G \in \mathcal{G}_\Lambda: \\ T(G) = \mathtt{t}}} \prod_{e \in E(G)} (e^{-\beta w_e^+} - 1) \prod_{i \in V(G)} e^{-\beta \mu_i^+} \,. \tag{5.15}$$

Isolated sites $\{i\} \subset \Lambda$ are also considered as trees. In this case, $\omega^+(\{i\}) = e^{-\beta \mu_i^+}$. The following lemma shows how the re-formulation in terms of trees allows to take advantage of the constraint.

Lemma 5.1. *Let $T \in \mathcal{T}_\Lambda$ be a forest such that $1(V(T)) = 1$. Then for each tree $\mathtt{t} \in T$,*

$$\|\omega^+(\mathtt{t})\|_{H_+} \leq \prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\frac{1}{4}\beta} \,. \tag{5.16}$$

Proof. For each $\mathtt{t} \in T$, let $E^*(\mathtt{t})$ denote the set of edges of the maximal connected graph of $\{G \in \mathcal{G}_\Lambda : T(G) = \mathtt{t}\}$ (see [Pf]). We can express the weight as follows:

$$\begin{aligned} \omega^+(\mathtt{t}) &= \prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\beta \mu_i^+} \sum_{\substack{G \in \mathcal{G}_\Lambda: \\ T(G) = \mathtt{t}}} \prod_{e \in E(G) \setminus E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \\ &= \prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\beta \mu_i^+} \prod_{e \in E^*(\mathtt{t}) \setminus E(\mathtt{t})} e^{-\beta w_e^+} \,. \end{aligned}$$

Since $1(V(T)) = 1$, the constraint (5.12) is satisfied, and the last product can be bounded by:

$$\prod_{e \in E^*(\mathtt{t}) \setminus E(\mathtt{t})} e^{\beta |w_e^+|} \leq \prod_{i \in V(\mathtt{t})} \prod_{\substack{e = \{i,j\} \\ j \in V(\mathtt{t})}} e^{\beta |w_e^+|} \tag{5.17}$$

$$= \prod_{i \in V(\mathtt{t})} \exp \beta \sum_{\substack{e = \{i,j\} \\ j \in V(\mathtt{t})}} |w_e^+| \leq \prod_{i \in V(\mathtt{t})} e^{\beta \delta} \,. \tag{5.18}$$

This gives the result, since $\operatorname{Re} \mu_i^+ \geq \frac{1}{2}$ by (5.10), and $\delta \leq 2^{-d} \leq \frac{1}{4}$. □

Notice that to obtain (5.18), we only needed that the bound

$$\sum_{\substack{e = \{i,j\} \\ j \in V(\mathtt{t})}} |w_e^+| \leq \delta \qquad \forall\, i \in V(\mathtt{t}) \tag{5.19}$$

be satisfied. This is weaker than (5.12) and clearly $1(V(T)) = 1$ only if (5.19) is satisfied for all $\mathtt{t} \in T$. In the sequel we can thus assume that the trees we consider always satisfy (5.19), independently of each other. So the bound (5.16) can always be used. A direct consequence of the last lemma is the following result, which shows that trees and their weights satisfy the main condition ensuring convergence of cluster expansions (see Appendix C for notations and main results on the cluster expansion).

Corollary 5.1. *Let $0 < c \le \frac{1}{8}\beta$, $\epsilon > 0$. There exists $\gamma_0 > 0$ and $\beta_1 = \beta_1(\epsilon)$ such that for all $\gamma \in (0, \gamma_0)$, $\beta \ge \beta_1$, the following bound holds:*

$$\sum_{\mathtt{t}:V(\mathtt{t})\ni 0} \|\omega^+(\mathtt{t})\|_{H_+} e^{c|V(\mathtt{t})|} \le \epsilon\,. \tag{5.20}$$

Proof. Using Lemma 5.1,

$$\|\omega^+(\mathtt{t})\|_{H_+} e^{c|V(\mathtt{t})|} \le \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{8}\beta}\,. \tag{5.21}$$

When $\mathtt{t}$ is a single isolated point (the origin), then we have a factor $e^{-\frac{1}{8}\beta}$. When $V(\mathtt{t}) \ni 0$, $E(\mathtt{t}) \ne \emptyset$, we define the generation of $\mathtt{t}$, $\mathrm{gen}(\mathtt{t})$, as the number of edges of the longest self avoiding path in $\mathtt{t}$ starting at the origin. The sum in (5.20) is bounded by

$$\begin{aligned}
e^{-\frac{1}{8}\beta} + &\sum_{g\ge 1} \sum_{\substack{\mathtt{t}:V(\mathtt{t})\ni 0\\ \mathrm{gen}(\mathtt{t})=g}} \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{8}\beta}\\
&\le e^{-\frac{1}{8}\beta} + \sum_{g\ge 1} e^{-\frac{1}{16}\beta g} \sum_{\substack{\mathtt{t}:V(\mathtt{t})\ni 0\\ \mathrm{gen}(\mathtt{t})=g}} \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})} e^{-\frac{1}{16}\beta}\\
&\le e^{-\frac{1}{8}\beta} + \sum_{g\ge 1} e^{-\frac{1}{16}\beta g} \alpha_g\,,
\end{aligned}$$

where we defined ($V_l(\mathtt{t})$ is the set of leaves of the tree $\mathtt{t}$):

$$\alpha_g := \sum_{\substack{\mathtt{t}:V(\mathtt{t})\ni 0\\ \mathrm{gen}(\mathtt{t})=g}} \prod_{e\in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i\in V(\mathtt{t})\setminus V_l(\mathtt{t})} e^{-\frac{1}{16}\beta} \prod_{i\in V_l(\mathtt{t})} e^{-\frac{1}{32}\beta} \tag{5.22}$$

We are going to show that $\alpha_{g+1} \le \alpha_g$ for all $g \ge 1$. Before going further, we define

$$\gamma_0 := \sup\left\{\gamma > 0 : 2c_\gamma \gamma^d \sup_s J(s) \le \tfrac{1}{64}\right\}. \tag{5.23}$$

Since $e^{-\beta w_e^+} - 1 \leq \beta |w_e^+| e^{\beta |w_e^+|}$ and $|w_e^+| = 2\phi_{ij}$ we can bound, when $\gamma \leq \gamma_0$,

$$\sum_{e \ni 0} \big(e^{-\beta w_e^+} - 1\big) e^{-\frac{1}{32}\beta} \leq \beta e^{-\frac{1}{64}\beta} \sum_{e \ni 0} |\omega_e^+| \leq 2\beta e^{-\frac{1}{64}\beta} \equiv \beta \zeta(\beta) \,. \tag{5.24}$$

Clearly, a tree $\mathtt{t}$ of generation $g+1$ can be obtained from a sub-tree $\mathtt{t}' \subset \mathtt{t}$ of generation g by attaching edges to leaves of $\mathtt{t}'$. Let x be a leaf of $\mathtt{t}'$. The sum over all possible edges (if any) attached at x is bounded by

$$1 + \sum_{k \geq 1} \frac{1}{k!} \sum_{e_1 \ni x} \cdots \sum_{e_k \ni x} \prod_{i=1}^{k} \big(e^{-\beta w_{e_i}^+} - 1\big) e^{-\frac{1}{32}\beta} \leq 1 + \sum_{k \geq 1} \frac{1}{k!} (\beta \zeta(\beta))^k = e^{\beta \zeta(\beta)} \,.$$

Assuming β is large enough so that $\zeta(\beta) \leq \frac{1}{32}$, the weight of the leaf x changes into $e^{-\frac{1}{16}\beta} e^{\beta\zeta(\beta)} \leq e^{-\frac{1}{32}\beta}$, which is exactly what appears in α_g. This shows that $\alpha_{g+1} \leq \alpha_g$. We then have $\alpha_{g+1} \leq \alpha_g \leq \cdots \leq \alpha_1$. Like we just did, it is easy to see that $\alpha_1 \leq e^{-\frac{1}{32}\beta}$. This proves the result. □

A sum over polymers. After the partial re-summation over the graphs having the same spanning tree, the constraint $1(V(T))$ in (5.14) still depends on the relative positions of the trees. This "multi-body interaction" can be worked out by expanding

$$1(V(T)) = \prod_{i \in [\Lambda]_R} 1_i(V(T)) = \prod_{i \in [\Lambda]_R} \big(1 + 1_i^c(V(T))\big) = \sum_{M \subset [\Lambda]_R} \prod_{i \in M} 1_i^c(V(T)) \,,$$

where $1_i^c(V(T)) := 1_i(V(T)) - 1$. This yields

$$\mathscr{Z}_r = \sum_{T \in \mathcal{T}_\Lambda} \sum_{M \subset [\Lambda]_R} \Big(\prod_{i \in M} 1_i^c(V(T))\Big)\Big(\prod_{\mathtt{t} \in T} \omega^+(\mathtt{t})\Big) \,. \tag{5.25}$$

Consider a pair (T, M) in (5.25). Let $i \in M$. The function $1_i^c(V(T))$ is non-zero only when i is not $(\delta, +)$-correct; it depends on the presence of trees of T in the R-neighbourhood of i and possibly on the points of $B(\eta_{\Lambda^c})$ if $B_R(i) \cap \Lambda^c \neq \emptyset$. To make these dependencies only local, we are going to link the R-neighbourhood of points of M with the trees of T.

Consider the graph $N = N(M)$ defined as follows. The vertices of N are given by

$$V(N) := \bigcup_{i \in M} B_R(i) \,, \tag{5.26}$$

and N has an edge between x and y if and only if $\langle x, y \rangle$ is a pair of nearest neighbours of the same box $B_R(i)$ for some $i \in M$. The graph N decomposes naturally into connected components (in the sense of graph theory) $N_1, N_2, \ldots, N_K$. Some of these components can intersect Λ^c.

We then link trees $\mathtt{t}_i \in T$ with components $N_j \in N$. To this end, we define an abstract graph $\hat{G}$: to each tree $\mathtt{t}_i \in T$, associate an abstract vertex w_i and to each component N_j an abstract vertex z_j. The edges of $\hat{G}$ are defined as follows: $\hat{G}$ has only edges between vertices w_i and z_j, and this occurs if and only if $V(\mathtt{t}_i) \cap V(N_j) \neq \emptyset$. Consider a connected component of $\hat{G}$, whose vertices $\{w_{i_1}, \ldots, w_{i_l}, z_{j_1}, \ldots, z_{j_l}\}$ correspond to a set $P'_l = \{\mathtt{t}_{i_1}, \ldots, \mathtt{t}_{i_l}, N_{j_1}, \ldots, N_{j_l}\}$. We change P'_l into a set P_l, using the following decimation procedure: if $P'_l = \{\mathtt{t}_{i_1}\}$ is a single tree then $P_l := P'_l$. Otherwise,
1) delete from P'_l all trees $\mathtt{t}_{i_k}$ that have no edges,
2) for all tree $\mathtt{t}_{i_k}$ containing at least one edge, delete all edges $e \in E(\mathtt{t}_{i_k})$ whose both end-points lie in the same component N_{j_m}.
The resulting set is of the form $P_l = \{\mathtt{t}_{s_1}, \ldots, \mathtt{t}_{s_l}, N_{j_1}, \ldots, N_{j_l}\}$, where each tree $\mathtt{t}_{s_i}$ is a sub-tree of one of the trees $\{\mathtt{t}_{i_1}, \ldots, \mathtt{t}_{i_l}\}$. P_l is called a polymer. The decimation procedure $P'_l \Rightarrow P_l$ is depicted on Figure 5.1.

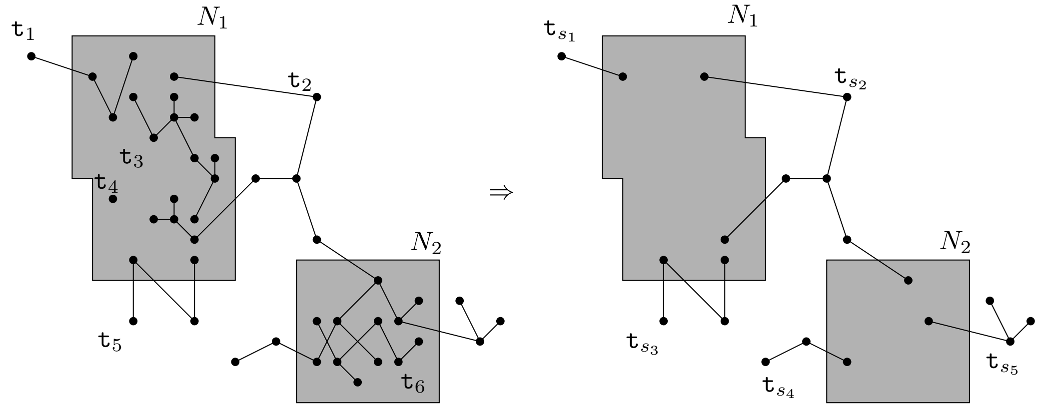

Figure 5.1: The decimation procedure $P'_l \Rightarrow P_l$. The hatched polygons represent the body $\mathcal{B}(P_l)$ and the legs are the trees $\{\mathtt{t}_{s_1}, \mathtt{t}_{s_2}, \mathtt{t}_{s_3}, \mathtt{t}_{s_4}, \mathtt{t}_{s_5}\}$. Each $\mathtt{t}_{s_j}$ is a sub-tree of some $\mathtt{t}_i$.

The body of P_l is $\mathcal{B}(P_l) := V(N_{j_1}) \cup \cdots \cup V(N_{j_l})$. The legs of P_l, $\mathcal{L}(P_l)$, are the trees $\{\mathtt{t}_{s_1}, \ldots, \mathtt{t}_{s_l}\}$.

A polymer can have no body (in which case it is a tree of $\mathcal{T}_\Lambda$), or no legs (in which case it is a single component N_{j_1}). We define the support $V(P)$ as the total set of sites:

$$V(P) := \bigcup_{\mathtt{t}\in\mathcal{L}(P)} V(\mathtt{t}) \cup \bigcup_i V(N_i)\,. \tag{5.27}$$

Often we denote $V(P)$ also by P. Two polymers are compatible if and only if $V(P_1) \cap V(P_2) = \emptyset$, denoted $P_1 \sim P_2$. We have thus associated to each pair (T, M) a family of pairwise compatible polymers $\{P\} := \varphi(T, M)$. The set of all possible polymers constructed in this way is denoted $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$. The representation of $\mathcal{Z}_r$ in terms of polymers is then

$$\mathcal{Z}_r = \sum_{\substack{\{P\}\subset\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})\\ \text{compat.}}} \prod_{P\in\{P\}} \omega^+(P)\,, \tag{5.28}$$

where the weight of a polymer P is defined by

$$\omega^+(P) := \sum_{\substack{(T,M):\\ \varphi(T,M)=P}} \Big(\prod_{i\in M} 1^c_i(V(T))\Big)\Big(\prod_{\mathtt{t}\in T}\omega^+(\mathtt{t})\Big)\,. \tag{5.29}$$

We should have in mind that $\omega^+(P)$ depends on the position of P inside the volume Λ, via the boundary condition η_{Λ^c}: more precisely if $\mathcal{B}(P) \cap \Lambda^c \neq \emptyset$ or if there exists a leg $\mathtt{t} \in \mathcal{L}(P)$ such that $d(\mathtt{t}, \Lambda^c) \leq R$. Therefore, we define the family $\mathcal{P}^+$ of free polymers of type $+$ whose weights depends only on the intrinsic structure of P, and not on the boundary condition. The family $\mathcal{P}^+$ is translation invariant, as well as the weight of each of its polymers. To any finite family $\mathcal{P}$, we associate the partition function

$$\mathcal{Z}_r(\mathcal{P}) := \sum_{\substack{\{P\}\subset\mathcal{P}\\ \text{compat.}}} \prod_{P\in\{P\}} \omega^+(P)\,, \tag{5.30}$$

where the product equals 1 when $\{P\} = \emptyset$. For instance,

$$Z_r{}^+(\Lambda; \eta_{\Lambda^c}) = e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\,. \tag{5.31}$$

Everything we have done until now can be done for a $-$-admissible boundary condition τ_{Λ^c}, yielding a family of polymers $\mathcal{P}^-_\Lambda(\tau_{\Lambda^c})$, with weights $\omega^-(P)$. In this case, sites get a factor $e^{-\beta\mu^-_i}$. In particular, if we consider the spin-flipped boundary condition $-\eta_{\Lambda^c}$ defined by $(-\eta_{\Lambda^c})_i := -(\eta_{\Lambda^c})_i$, which is $-$-admissible, we have when h is purely imaginary [2],

$$\overline{\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))} = \mathcal{Z}_r(\mathcal{P}^-_\Lambda(-\eta_{\Lambda^c}))\,. \tag{5.32}$$

[2] Here, $\overline{z}$ denotes the complex conjugate of z.

5.2 Analyticity of the Restricted Phases

Define the restricted pressure by

$$\mathfrak{p}^+_{r,\gamma} := \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{1}{\beta|\Lambda|} \log Z_r{}^+(\Lambda; +_{\Lambda^c}) \ , \tag{5.33}$$

where the thermodynamic limit is taken along a sequence of cubes. A result of the present section is that the restricted pressure, unlike the total pressure $\mathfrak{p}_\gamma$, behaves analytically at $h = 0$.

The point is that we linked trees with the R-neighbourhood of points of the set M, and we must now see that this thickening does not destroy, from the point of view of entropy, the uniformity we have been able to obtain with respect to the scaling parameter γ. Moreover, the body of polymers can intersect Λ^c. At this point we will see that $\delta - \tilde{\delta} > 0$ is crucial. We study the weight $\omega^+(P)$ ($\omega^-(P)$ is similar by symmetry).

Lemma 5.2. *There exists β_2 and $\tau_0 > 0$ such that for all $\beta \geq \beta_2$ and for all $\gamma \in (0, \gamma_0)$, the following holds: each polymer $P \in \mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ satisfies*

$$\|\omega^+(P)\|_{H_+} \leq e^{-\tau_0 \beta |\mathcal{B}(P)|} \prod_{e \in \mathcal{L}(P)} (e^{-\beta w_e^+} - 1) \prod_{i \in \mathcal{L}(P)} e^{-\frac{1}{12}\beta} \,. \tag{5.34}$$

Proof. Remember that the bound (5.16) holds for each tree under consideration. If $\mathcal{B}(P) = \emptyset$, then P is a tree and the result follows from Lemma 5.1. Otherwise, $\|\omega^+(P)\|_{H_+}$ is bounded by

$$\sum_{\substack{(T,M): \\ \varphi(T,M)=P}} \Big(\prod_{i \in M} |1^c_i(V(T))| \Big) \prod_{\mathtt{t} \in T} \Big(\prod_{e \in E(\mathtt{t})} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t})} e^{-\frac{1}{4}\beta} \Big) \,.$$

Consider a pair (T, M) such that $\varphi(T, M) = P$. Let $i_0 \in M$, and assume $1^c_{i_0}(V(T)) \neq 0$. This implies, with regard to (5.12),

$$\sum_{\substack{e=\{i_0,j\} \\ j \in V(T) \cup B}} |w_e^+| > \delta \,. \tag{5.35}$$

But, according to (5.13), we have

$$\sum_{\substack{e=\{i_0,j\} \\ j \in B}} |w_e^+| \leq \tilde{\delta} \,. \tag{5.36}$$

This implies the crucial lower bound

$$\sum_{\substack{e=\{i_0,j\} \\ j\in V(T)}} |w_e^+| \geq \delta - \tilde{\delta} > 0\,. \tag{5.37}$$

Since $|w_e^+| = 2\phi_{ij} \leq 2c_\gamma \gamma^d \sup_s J(s)$, we can find a constant c_3 such that

$$|V(T) \cap B_R^\bullet(i_0)| > (\delta - \tilde{\delta})c_3|B_R(i_0)|\,. \tag{5.38}$$

In this sense, the forests that contribute to $\omega^+(P)$ accumulate in the neighbourhood of each point $i_0 \in M$. See Figure 5.2. Let M_0 be any $2R$-approximant of

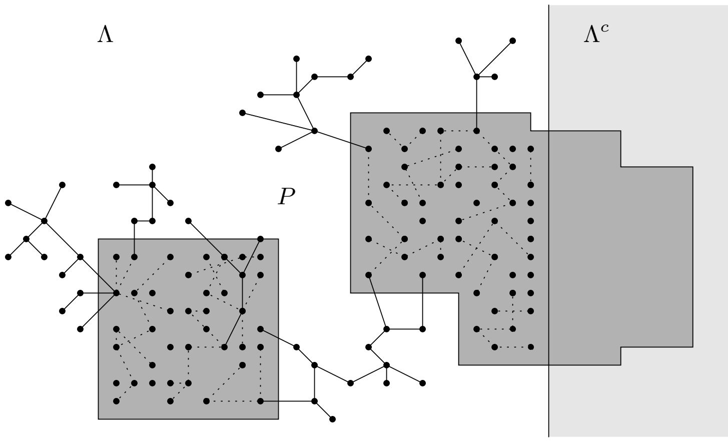

Figure 5.2: The re-summation of Lemma 5.2. We emphasised the fact that the forest T must have many points in $\mathcal{B}(P) \cap \Lambda$, as was shown in (5.39).

M. Then we have $|\mathcal{B}(P)| \leq |M_0||B_{3R}(0)|$ and so

$$|V(T) \cap \mathcal{B}(P)| \geq \sum_{i_0 \in M_0} |V(T) \cap B_R(i_0)| \geq (\delta - \tilde{\delta})c_4|\mathcal{B}(P)| \tag{5.39}$$

where c_4 is a constant. Now, each $i \in V(T)$ gets a factor $e^{-\frac{1}{4}\beta} = e^{-3\frac{1}{12}\beta}$. One factor $e^{-\frac{1}{12}\beta}$ contributes to extract a term decreasing exponentially fast with the size of $\mathcal{B}(P)$, using (5.39):

$$e^{-\frac{1}{12}(\delta-\tilde{\delta})c_4\beta|\mathcal{B}(P)|}\,. \tag{5.40}$$

A second factor $e^{-\frac{1}{12}\beta}$ contributes to the weight of the legs. Extracting this contribution gives

$$\prod_{e\in\mathcal{L}(P)} (e^{-\beta w_e^+} - 1) \prod_{i\in\mathcal{L}(P)} e^{-\frac{1}{12}\beta}\,, \tag{5.41}$$

The last factor $e^{-\frac{1}{12}\beta}$ is used to re-sum over all the possible configurations of T inside the body $\mathcal{B}(P)$ (see Figure 5.2), that is over all forests T', $V(T') \subset \mathcal{B}(P)$, where each tree $\mathtt{t}' \in T'$ gets a weight bounded by

$$\omega_0(\mathtt{t}') := \prod_{e \in E(\mathtt{t}')} (e^{-\beta w_e^+} - 1) \prod_{i \in V(\mathtt{t}')} e^{-\frac{1}{12}\beta} . \tag{5.42}$$

The remaining sum is thus bounded by

$$\sum_{T' : V(T') \subset \mathcal{B}(P)} \prod_{\mathtt{t}' \in T'} \omega_0(\mathtt{t}') \equiv \Theta_0(\mathcal{B}(P)) . \tag{5.43}$$

This partition function can be studied with a convergent cluster expansion (see Appendix C). Proceeding as we did in Corollary 5.1, we can take β sufficiently large so that the weight $\omega_0(\mathtt{t}')$ satisfies (5.20). We can then guarantee that

$$|\Theta_0(\mathcal{B}(P))| \leq e^{|\mathcal{B}(P)|} . \tag{5.44}$$

The sum over all possible sets M such that $N(M)$ has a set of vertices given by $\mathcal{B}(P)$ is bounded by $2^{|\mathcal{B}(P)|}$. Altogether these bounds give

$$e^{-\frac{1}{12}(\delta - \tilde{\delta})c_4 \beta |\mathcal{B}(P)|} 2^{|\mathcal{B}(P)|} e^{|\mathcal{B}(P)|} \equiv e^{-\tau_0 \beta |\mathcal{B}(P)|} ,$$

which finishes the proof. □

We now give the consequence of this lemma, namely that polymers satisfy the main criterion needed for having a convergent cluster expansion.

Corollary 5.2. *Let* $0 < c \leq \min(\frac{\tau_0}{2}, \frac{1}{24})\beta$, $\epsilon > 0$. *There exists* $\beta_3 = \beta_3(\epsilon)$, *such that for all* $\beta \geq \beta_3$ *and for all* $\gamma \in (0, \gamma_0)$, *the following holds:*

$$\sum_{P : V(P) \ni 0} \|\omega^+(P)\|_{H_+} e^{c|V(P)|} \leq \epsilon . \tag{5.45}$$

Proof. Lemma 5.2 allows to bound

$$\|\omega^+(P)\|_{H_+} \leq \Big(\prod_{N \in P} \omega_0(N) \Big) \Big(\prod_{\mathtt{t} \in \mathcal{L}(P)} \omega_0(\mathtt{t}) \Big) \equiv \omega_0(P) , \tag{5.46}$$

where the weight of each component of the body N is $\omega_0(N) := e^{-\tau_0 \beta |V(N)|}$ and the weight of each leg $\mathtt{t}$ was defined in (5.42). Fix $\epsilon > 0$ small. It is easy to show that when β is large enough,

$$\sum_{N : V(N) \ni 0} \omega_0(N) e^{(c+\epsilon)|V(N)|} \leq \tfrac{1}{2}\epsilon , \tag{5.47}$$

and, proceeding like in Corollary 5.1,

$$\sum_{\mathtt{t}:V(\mathtt{t})\ni 0} \omega_0(\mathtt{t}) e^{(c+\epsilon)|V(\mathtt{t})|} \leq \tfrac{1}{2}\epsilon\,. \tag{5.48}$$

Let $n(P)$ denote the number of objects (components N and trees $\mathtt{t}$) contained in P. That is, if $P = \{\mathtt{t}_1, \dots, \mathtt{t}_L, N_1, \dots, N_K\}$, then $n(P) = L + K$. We will show by induction on $N = 1, 2, \dots$ that

$$\lambda_N := \sum_{\substack{P:V(P)\ni 0\\ n(P)\leq N}} \omega_0(P) e^{c|V(P)|} \leq \epsilon\,, \tag{5.49}$$

which will finish the proof. If $N = 1$ then P can be either a single component N or a tree $\mathtt{t}$. The bound then follows from (5.47) and (5.48). Suppose β is large and that the bound holds for N. If P satisfies $V(P) \ni 0$, $n(P) \leq N+1$, we choose an object of P that contains the origin (which can be a tree $\mathtt{t}_0$ or a component N_0), and decompose P as follows: either $P = \{N_0\} \cup \{P_1, \dots, P_k\}$ with $V(N_0) \ni 0$, $V(P_i) \cap V(N_0) \neq \emptyset$, $n(P_i) \leq N$, $P_i \sim P_j$ for $i \neq j$, or $P = \{\mathtt{t}_0\} \cup \{P_1, \dots, P_k\}$ with $V(\mathtt{t}_0) \ni 0$, and $V(P_i) \cap V(\mathtt{t}_0) \neq \emptyset$, $n(P_i) \leq N$, $P_i \sim P_j$ for $i \neq j$. In the first case, we have, using the induction hypothesis and (5.47),

$$\sum_{N_0:V(N_0)\ni 0} \omega_0(N_0) e^{c|V(N_0)|} \sum_{k\geq 0} \frac{1}{k!} \Big(\sum_{\substack{P:V(P)\cap V(N_0)\neq\emptyset\\ n(P)\leq N}} \omega_0(P) e^{c|V(P)|} \Big)^k \tag{5.50}$$

$$\leq \sum_{N_0:V(N_0)\ni 0} \omega_0(N_0) e^{c|V(N_0)|} \sum_{k\geq 0} \frac{1}{k!} \big(|V(N_0)|\lambda_N\big)^k \tag{5.51}$$

$$\leq \sum_{N_0:V(N_0)\ni 0} \omega_0(N_0) e^{c|V(N_0)|} e^{\epsilon|V(N_0)|} \leq \tfrac{1}{2}\epsilon\,. \tag{5.52}$$

In the second case the same computation yields, using (5.48),

$$\sum_{\mathtt{t}_0:V(\mathtt{t}_0)\ni 0} \omega_0(\mathtt{t}_0) e^{c|V(\mathtt{t}_0)|} \sum_{k\geq 0} \frac{1}{k!} \Big(\sum_{\substack{P:V(P)\cap V(\mathtt{t}_0)\neq\emptyset\\ n(P)\leq N}} \omega_0(P) e^{c|V(P)|} \Big)^k$$

$$\leq \sum_{\mathtt{t}_0:V(\mathtt{t}_0)\ni 0} \omega_0(\mathtt{t}_0) e^{c|V(\mathtt{t}_0)|} e^{\epsilon|V(\mathtt{t}_0)|} \leq \tfrac{1}{2}\epsilon\,. \tag{5.53}$$

This shows that $\lambda_{N+1} \leq \epsilon$ and finishes the proof. □

We now state the main result concerning restricted phases and their analyticity properties. We refer to Appendix C for notations. Here polymers play the role of animals. We know that if two polymers P_1, P_2 are incompatible then $V(P_1) \cap V(P_2) \neq \emptyset$. Clusters of polymers associated to $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ are denoted $\hat{P} \in \hat{\mathcal{P}}^+_\Lambda(\eta_{\Lambda^c})$.

The bound (5.45) implies that the main condition (C.4) is satisfied. By Lemma C.1, we have

$$\sup_{x\in\Lambda}\sum_{\hat{P}\ni x}\|\omega^+(\hat{P})\|_{H_+}\leq\sup_{x\in\Lambda}\sum_{\hat{P}\ni x}|\omega_0(\hat{P})|\leq\eta(\epsilon)\,, \tag{5.54}$$

where the weights $\omega^+(\hat{P})$ and $\omega_0(\hat{P})$ are defined like in (C.3). Since ϵ can be made arbitrarily small by taking β large enough, we will replace $\eta(\epsilon)$ by a function $\epsilon_r(\beta)$, where the subscript r is to indicate that this function concerns the restricted phase. We define $\tilde{H}_+ := \{\mathrm{Re}\, h > -\frac{1}{16}\}\subset H_+$.

Theorem 5.1. *Let β be large enough, $\gamma\in(0,\gamma_0)$. Let $\Lambda\in\mathcal{C}^{(l)}$ and η_{Λ^c} be a $+$-admissible boundary condition. Then $\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))$ has a cluster expansion that converges normally in H_+, given by*

$$\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))=\sum_{\hat{P}\in\hat{\mathcal{P}}^+_\Lambda(\eta_{\Lambda^c})}\omega^+(\hat{P})\,. \tag{5.55}$$

The maps $h\mapsto\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))$, $h\mapsto\mathfrak{p}^+_{r,\gamma}(h)$ are analytic in H_+. Moreover there exists a function $\epsilon_r(\beta)$, $\lim_{\beta\nearrow\infty}\epsilon_r(\beta)=0$, such that

$$\big\|\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\big\|_{H_+}\leq\epsilon_r(\beta)|\Lambda|\,,\qquad\sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+_\Lambda(\eta_{\Lambda^c})\\ \hat{P}\ni 0}}\|\omega^+(\hat{P})\|_{H_+}\leq\epsilon_r(\beta)\,,$$

$$\big\|\frac{\mathrm{d}}{\mathrm{d}h}\log\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\big\|_{\tilde{H}_+}\leq\epsilon_r(\beta)|\Lambda|\,. \tag{5.56}$$

Finally,

$$Z_r{}^+(\Lambda;\eta_{\Lambda^c})=e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))=\exp\big(\beta\mathfrak{p}^+_{r,\gamma}|\Lambda|+\triangle^+_r(\Lambda)\big)\,, \tag{5.57}$$

with $\|\triangle^+_r(\Lambda)\|_{H_+}\leq\epsilon_r(\beta)|\partial^+_R\Lambda|$.

The proof of the theorem follows easily from Lemma C.1. Analyticity follows from the fact that the convergence is normal on H_+. The bound on the first derivative is obtained by using the Cauchy formula: any disc of radius $\frac{1}{16}$ centered at $z\in\tilde{H}_+$ is contained in H_+. The same can be done for large order derivatives: there exists a constant $C_r>0$ such that for all integer $k\geq 2$,

$$\frac{1}{|\Lambda|}\left|\frac{\mathrm{d}^k}{\mathrm{d}h^k}\log Z_r{}^+(\Lambda;\eta_{\Lambda^c})\right|_{h=0}\leq C_r^k k!\,,\qquad|\mathfrak{p}^{+(k)}_{r,\gamma}(0)|\leq C_r^k k!\,. \tag{5.58}$$

The last statement of the theorem follows by the usual rearrangment of the terms of the cluster expansion.

Chapter 6

The Phase Diagram

Until now we have a notion of contour, with a Peierls constant uniform in the scaling parameter, and a precise description of restricted phases in terms of graphs. We must now bound these objects together to study the phase diagram of large systems in a neighbourhood of $h = 0$. Throughout this section we assume $\gamma \in (0, \gamma_0)$ is fixed, where γ_0 was defined during the analysis of the restricted phases, in (5.23).

Consider the partition function

$$Z^+(\Lambda) := \sum_{\sigma_\Lambda \in \Omega_\Lambda^+} e^{-\beta H_\Lambda(\sigma_\Lambda +_{\Lambda^c})}\,, \tag{6.1}$$

where

$$\Omega_\Lambda^+ := \{\sigma_\Lambda \in \Omega_\Lambda : d(I^*(\sigma_\Lambda +_{\Lambda^c}), \Lambda^c) > l\}\,. \tag{6.2}$$

For each $\sigma_\Lambda \in \Omega_\Lambda^+$, the decomposition of $I^*(\sigma_\Lambda +_{\Lambda^c})$ into connected components yields an admissible family $\{\Gamma\}$, such that $\Gamma \subset \Lambda$ and $d(\Gamma, \Lambda^c) > l$ for each $\Gamma \in \{\Gamma\}$. Then, Λ is decomposed into $\Lambda = \{\Gamma\} \cup \Lambda^+ \cup \Lambda^-$, where $\Lambda^\#$ are the points of $\Lambda \backslash \{\Gamma\}$ that are $(\delta, \#)$-correct for the configuration $\sigma_\Lambda +_{\Lambda^c}$.

In (6.1), we re-sum over the configurations σ_{Λ^+} (resp. σ_{Λ^+}) on Λ^+ (resp. Λ^-) that yield the same set of contours $\{\Gamma\}$. In Proposition 4.1 we characterised explicitly the constraints satisfied by the configurations $\sigma_{\Lambda^\pm}$: each point $i \in [\Lambda^+]_R$ must be $(\delta, +)$-correct for the configuration $\sigma_{\Lambda^+} +_{\Lambda^c} \sigma_{\{\Gamma\}}$, where $\sigma_{\{\Gamma\}}$ is the configuration specified by the contours on the union of their supports. Similarly, each point $i \in [\Lambda^-]_R$ must be $(\delta, -)$-correct for the configuration $\sigma_{\Lambda^-}\sigma_{\{\Gamma\}}$. Using the reformulation of the hamiltonian given in Lemma 4.6 we get:

$$Z^+(\Lambda) = \sum_{\{\Gamma\} \subset \Lambda} \Big(\prod_{\Gamma \in \{\Gamma\}} \rho(\Gamma) \Big) Z_r{}^+(\Lambda^+; +_{\Lambda^c}\sigma_{\{\Gamma\}}) Z_r{}^-(\Lambda^-; \sigma_{\{\Gamma\}})\,, \tag{6.3}$$

where the sum is over admissible families of contours, and

$$\rho(\Gamma) := e^{-\beta H_\Gamma(\sigma[\Gamma])} . \tag{6.4}$$

Notice that when $\{\Gamma\} = \emptyset$, then $\Lambda \equiv \Lambda^+$ and the summand of (6.3) equals ${Z_r}^+(\Lambda; +_{\Lambda^c})$.

Since they are subject to boundary conditions that depend on the family of contours $\{\Gamma\}$, the restricted phases induce interactions between the contours. Nevertheless, the boundary conditions imposed by the contours and $+_{\Lambda^c}$ on Λ^+ and Λ^- are admissible (in the sense of Definition 5.1). This implies that the results of Chapter 5 can be used for the restricted partition functions appearing in (6.3). As a consequence, the interactions created by the polymers of the restricted phases decay exponentially fast with the distance.

Since we need to represent the partition function with objects whose compatibility is purely geometrical, we need to proceed by induction, and consider systems living in the interior of external contours. Therefore, we must study functions similar to (6.3), with an arbitrary +-admissible boundary condition η_{Λ^c}. We thus define

$$\Theta^+(\Lambda; \eta_{\Lambda^c}) := \sum_{\{\Gamma\}\subset\Lambda} \Big(\prod_{\Gamma\in\{\Gamma\}} \rho(\Gamma)\Big) {Z_r}^+(\Lambda^+; \eta_{\Lambda^c}\sigma_{\{\Gamma\}}) {Z_r}^-(\Lambda^-; \sigma_{\{\Gamma\}}) , \tag{6.5}$$

Contours always lie at least at distance l from Λ^c. The external contours of $\{\Gamma\}$ can be subject to particular constraints, but we omit it in the notation. Notice that for the empty family $\{\Gamma\} = \emptyset$, the summand corresponds to a pure restricted phase ${Z_r}^+(\Lambda; \eta_{\Lambda^c})$.

The aim, in the study of $\Theta^+(\Lambda; \eta_{\Lambda^c})$, is to extract from (6.5) a global contribution of the restricted phase. In the Ising model, the same operation amounts to extract the trivial term $e^{\beta h|\Lambda|}$. Here we extract ${Z_r}^+(\Lambda, \eta_{\Lambda^c}) = e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))$, and our aim is to reach the representation (6.18). The deviations from the restricted phase will be described by *chains*, i.e. contours linked by clusters of polymers. In Section 6.1, we expose this linking procedure. In Section 6.2 we show how to handle the entropy of chains, preserving the uniformity in the scaling parameter γ. In Section 6.3 we study the weights of chains and their dependence on the magnetic field near $\{\mathrm{Re}\, h = 0\}$, i.e. at coexistence. In Section 6.4 we study pure phases, i.e. $\{\mathrm{Re}\, h > 0\}$ and $\{\mathrm{Re}\, h < 0\}$.

6.1 The Linking Procedure

We first express $\Theta^+(\Lambda;\eta_{\Lambda^c})$ as a sum over external contours. By Lemma 4.5, each external contour is of type $+$. Let $\{\Gamma\}$ be a family of external contours. Then, Λ is decomposed into

$$\Lambda = \mathrm{ext}_\Lambda\{\Gamma\} \cup \{\Gamma\} \cup \bigcup_{\Gamma\in\{\Gamma\}} \mathrm{int}\Gamma\,,$$

where $\mathrm{ext}_\Lambda\{\Gamma\} := \Lambda \cap \bigcap_{\Gamma\in\{\Gamma\}} \mathrm{ext}\Gamma$. For each family of admissible external contours $\{\Gamma\}$, we re-sum over the configurations whose external contours are given exactly by $\{\Gamma\}$. This induces, for all Γ, a partition function $\Theta^-(\mathrm{int}\Gamma;+\sigma_\Gamma)$, which can be expressed as in (6.5). On $\mathrm{ext}_\Lambda\{\Gamma\}$, we get a restricted partition function ${Z_r}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}})$. We thus have

$$\begin{aligned}&\Theta^+(\Lambda;\eta_{\Lambda^c}) = \\ &\quad {Z_r}^+(\Lambda;\eta_{\Lambda^c}) + \sum_{\substack{\{\Gamma\}\subset\Lambda\\ \text{ext.}}} {Z_r}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}}) \prod_\Gamma \rho(\Gamma)\Theta^-(\mathrm{int}\Gamma;\sigma_\Gamma)\,,\end{aligned} \tag{6.6}$$

where the sum is over non-empty families of external contours. Consider the configuration $-\sigma_\Gamma$ obtained by spin-flipping σ_Γ, i.e. $(-\sigma_\Gamma)_i := -(\sigma_\Gamma)_i$ for all $i\in\Gamma$. We introduce the functions ${Z_r}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)$ and $\Theta^+(\mathrm{int}\Gamma;-\sigma_\Gamma)$ and consider, for a while, the ratio

$$\frac{{Z_r}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}})\prod_\Gamma {Z_r}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}{{Z_r}^+(\Lambda;\eta_{\Lambda^c})}\,. \tag{6.7}$$

Using the polymer representation of Chapter 5, we consider the family of polymers $\mathcal{P}^+_{\mathrm{ext}} := \mathcal{P}^+_{\mathrm{ext}_\Lambda\{\Gamma\}}(\eta_{\Lambda^c}\sigma_{\{\Gamma\}})$ associated to ${Z_r}^+(\mathrm{ext}_\Lambda\{\Gamma\};\eta_{\Lambda^c}\sigma_{\{\Gamma\}})$, the families $\mathcal{P}^+_{\mathrm{int}\Gamma} := \mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma)$ associated to each of the ${Z_r}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)$, as well as the family $\mathcal{P}^+_\Lambda := \mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ associated to ${Z_r}^+(\Lambda;\eta_{\Lambda^c})$. Since the expansions of these functions are absolutely convergent, we can rearrange the terms. The volume contributions from $\mathrm{ext}_\Lambda\{\Gamma\}$ and $\bigcup_\Gamma \mathrm{int}\Gamma$ cancel, and we get

$$\frac{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{ext}})\prod_\Gamma \mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^+_\Lambda)} = \exp\Big(\sum_{\hat P} \pm\omega^+(\hat P) + \sum_\Gamma E^+_\Gamma\Big)\,,$$

where we used the abbreviation

$$\sum_{\hat P} \pm\omega^+(\hat P) \equiv \sum_{\substack{\hat P\in\hat{\mathcal{P}}^+_{\mathrm{ext}}\\ d(\hat P,\{\Gamma\})\le R}} \omega^+(\hat P) - \sum_{\substack{\hat P\in\hat{\mathcal{P}}^+_\Lambda\\ d(\hat P,\{\Gamma\})\le R\\ \hat P\cap\mathrm{ext}_\Lambda\{\Gamma\}\neq\emptyset}} \omega^+(\hat P)\,. \tag{6.8}$$

The sign $\pm$ in front of $\omega^+(\hat{P})$ is chosen in function of the sum to which $\hat{P}$ belongs. Define $\lambda^+(\hat{P}) := e^{\pm\omega^+(\hat{P})} - 1$ and expand

$$e^{\sum_{\hat{P}} \pm\omega^+(\hat{P})} = \prod_{\hat{P}} (1 + \lambda^+(\hat{P})) = \sum_{\{\hat{P}_1,\dots,\hat{P}_n\}} \prod_{i=1}^{n} \lambda^+(\hat{P}_i)\,. \tag{6.9}$$

The function E_Γ^+ depends only on the structure of Γ, and is given by

$$E_\Gamma^+ = \sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+_{\mathrm{int}\Gamma}\\ d(\hat{P},\Gamma)\le R}} \omega^+(\hat{P}) - \sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+\\ \hat{P}\cap\mathrm{ext}\Gamma=\emptyset\\ d(\hat{P},\Gamma)\le R}} \omega^+(\hat{P})\,, \tag{6.10}$$

where $\hat{\mathcal{P}}^+$ denotes the family of clusters associated to free polymers of type $+$. Notice that E_Γ^+ is analytic in H_+. Since $|[\Gamma]_R| \le 3^d|\Gamma|$ we have, if β is large enough (see Theorem 5.1)

$$\|E_\Gamma^+\|_{H_+} \le \frac{1}{3}|\Gamma|\,, \quad \|\frac{\mathrm{d}}{\mathrm{d}h}E_\Gamma^+\|_{\tilde{H}_+} \le \frac{1}{3}|\Gamma|\,. \tag{6.11}$$

If we define the weight (we denote $+\sigma_\Gamma \equiv \sigma_\Gamma$)

$$\omega^+(\Gamma) := \rho_1(\Gamma)\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\,, \tag{6.12}$$

where

$$\rho_1(\Gamma) := \rho(\Gamma)e^{-\beta h|\Gamma|}e^{E_\Gamma^+}\,, \tag{6.13}$$

we have

$$\frac{\Theta^+(\Lambda;\eta_{\Lambda^c})}{Z_r{}^+(\Lambda;\eta_{\Lambda^c})} = 1 + \sum_{\substack{\{\Gamma\}\subset\Lambda\\ \text{ext.}}} \sum_{\{\hat{P}_1,\dots,\hat{P}_n\}} \Big(\prod_{i=1}^{n}\lambda^+(\hat{P}_i)\Big)\Big(\prod_{\Gamma}\omega^+(\Gamma)\frac{\Theta^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}{Z_r{}^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}\Big)\,.$$

We can then repeat the same procedure of summing inside external contours of $\Theta^+(\mathrm{int}\Gamma;-\sigma_\Gamma)$, etc. This procedure continues until we reach contours whose interior can't contain any contour. At the end,

$$\frac{\Theta^+(\Lambda;\eta_{\Lambda^c})}{Z_r{}^+(\Lambda;\eta_{\Lambda^c})} = 1 + \sum_{\{\Gamma\}\subset\Lambda} \sum_{\{\hat{P}\}} \Big(\prod_{\hat{P}}\lambda^+(\hat{P})\Big)\Big(\prod_{\Gamma}\omega^+(\Gamma)\Big)\,, \tag{6.14}$$

where the sum over $\{\Gamma\} \subset \Lambda$ contains contours of type $+$, and each cluster $\hat{P}$ lies at distance at most R from one or several contours of $\{\Gamma\}$. For this reason, the weight of polymers can depend on the configuration σ_Γ of the contours Γ

that lie in their neighbourhood (or on η_{Λ^c}). We get rid of these dependencies by linking polymers to contours. Like we did in Section 5.1 (when linking trees with components of the graph N), we associate to each pair $(\{\Gamma\},\{\hat{P}\})$ an abstract graph $\hat{G}$ as follows: each contour $\Gamma_j \in \{\Gamma\}$ is represented by an abstract vertex z_j, each cluster $\hat{P}_k \in \{\hat{P}\}$ is represented by an abstract vertex w_k. This defines $V(\hat{G})$. Then, we put an edge between z_j and w_k if and only if $d(\Gamma_j, \hat{P}_k) \leq R$. We also put an edge between w_{k_1} and w_{k_2} if and only if $V(\hat{P}_{k_1}) \cap V(\hat{P}_{k_2}) \neq \emptyset$.
Each connected component of $\hat{G}$, with vertices, say, $\{z_{j_1},\dots,z_{j_l},w_{k_1},\dots,w_{k_l}\}$, represents a subset of $\{\Gamma\}\cup\{\hat{P}\}$ given by $X=\{\Gamma_{j_1},\dots,\Gamma_{j_l},\hat{P}_{k_1},\dots,\hat{P}_{k_l}\}$. X is called a chain of contours, or simply a chain. We denote by $\{X\}$ the family of chains associated to the pair $(\{\Gamma\},\{\hat{P}\})$. The chains of $\{X\}$ are of type $+$, and pairwise compatible by definition. The support of X, also written X, denotes the union $\bigcup_{\Gamma\in X}\Gamma \cup \bigcup_{\hat{P}\in X}\hat{P}$. Notice that if two chains X, X' are not compatible, then $b(X)\cap b(X') \neq \emptyset$, where

$$b(X) := \bigcup_{\Gamma\in X}[\Gamma]_l \cup \bigcup_{\hat{P}\in X}\hat{P}\,. \tag{6.15}$$

The weight of a chain is defined by

$$\omega^+(X) := \Big(\prod_{\hat{P}\in X}\lambda^+(\hat{P})\Big)\Big(\prod_{\Gamma\in X}\omega^+(\Gamma)\Big)\,, \tag{6.16}$$

and depends only on the intrinsic structure of the chain X (except, maybe, if $d(X,\Lambda^c)\leq R$). The final representation of the partition function is thus

$$\Theta^+(\Lambda;\eta_{\Lambda^c}) = {Z_r}^+(\Lambda;\eta_{\Lambda^c})\sum_{\{X\}}\prod_{X\in\{X\}}\omega^+(X) \tag{6.17}$$

$$\equiv {Z_r}^+(\Lambda;\eta_{\Lambda^c})\Xi^+(\Lambda;\eta_{\Lambda^c})\,. \tag{6.18}$$

In (6.17), the product is defined to be equal to 1 when $\{X\}=\emptyset$. This last expression nicely expresses the fact that chains of contours describe deviations from a restricted phase. For the restricted phase, there corresponds a family $\mathcal{P}^+_\Lambda(\eta_{\Lambda^c})$ associated to ${Z_r}^+(\Lambda;\eta_{\Lambda^c})$. Similarly, there corresponds a family of chains $\mathcal{X}^+_\Lambda(\eta_{\Lambda^c})$ associated to $\Xi^+(\Lambda;\eta_{\Lambda^c})$. The partition function can be written in terms of these families as

$$\Theta^+(\Lambda;\eta_{\Lambda^c}) = e^{\beta h|\Lambda|}\mathcal{Z}_r(\mathcal{P}^+_\Lambda(\eta_{\Lambda^c}))\Xi(\mathcal{X}^+_\Lambda(\eta_{\Lambda^c}))\,. \tag{6.19}$$

By definition, $\Xi(\mathcal{X}^+_\Lambda(\eta_{\Lambda^c})) := 1$ when $\mathcal{X}^+_\Lambda(\eta_{\Lambda^c})=\emptyset$. Everything that was done until now can be applied also to the case where η_{Λ^c} is $-$-admissible, yielding chains of type $-$.

6.2 The Entropy of Chains

Before starting the analysis of the weights, we show how a priori bounds on the weights $\lambda^+(\hat{P})$ and $\omega^+(\Gamma)$ allow to handle the summation of weights of chains. In this section we assume that $|\lambda^+(\hat{P})| \le \lambda_0(\hat{P})$, $|\omega^+(\Gamma)| \le \rho_0(\Gamma)$, i.e.

$$|\omega^+(X)| \le \Big(\prod_{\hat{P} \in X} \lambda_0(\hat{P}) \Big) \Big(\prod_{\Gamma \in X} \rho_0(\Gamma) \Big) \equiv \omega_0(X) \,. \tag{6.20}$$

Convention: Now and in the sequel we will always use a subscript "0" in the weight of an object to specify that it depends only on the geometric structure of the object (as we did in (5.46), Section 5.2). That is, such weights will always be translation invariant. When a weight is defined for an object, we use the same letter for the weight of the clusters of such objects (see Appendix C).

The proof of the following lemma is essentially the same as the one of Corollary 5.2. We use the notations $|\hat{P}| := |\bigcup_{P \in \hat{P}} V(P)|$, $|X| := \sum_{\Gamma \in X} |\Gamma| + \sum_{\hat{P} \in X} |\hat{P}|$.

Lemma 6.1. *Let $c > 0$, $\epsilon > 0$, and assume the weights $\lambda_0(\hat{P})$, $\rho_0(\Gamma)$ satisfy the bounds*

$$\sum_{\hat{P} \ni 0} \lambda_0(\hat{P}) e^{(c+\epsilon(2^d+1))|\hat{P}|} \le \frac{\epsilon}{2} \,, \qquad \sum_{\Gamma : [\Gamma]_l \ni 0} \rho_0(\Gamma) e^{(c+\epsilon)|[\Gamma]_l|} \le \frac{\epsilon}{2} \,. \tag{6.21}$$

Then the weight $\omega_0(X)$ satisfies the condition (C.4) *of Lemma C.1. Namely,*

$$\sum_{X : b(X) \ni 0} \omega_0(X) e^{c|b(X)|} \le \epsilon \,. \tag{6.22}$$

Proof. For a chain $X = \{\Gamma_1, \dots, \Gamma_L, \hat{P}_1, \dots, \hat{P}_M\}$, let $n(X) := L + M$ denote the number of objects composing X (a cluster $\hat{P}_i$ is considered as a single object). We show by induction on $N = 1, 2, \dots$ that [1]

$$\xi_N := \sum_{\substack{X : b(X) \ni 0 \\ n(X) \le N}} \omega_0(X) e^{c|b(X)|} \le \epsilon \,. \tag{6.23}$$

If $n(X) = 1$ then X contains a single object, i.e. a contour. Then $\xi_1 \le \epsilon$ follows from (6.21). So suppose (6.23) holds for N, and consider ξ_{N+1}; this sum can be bounded by a sum in which each chain X is decomposed into $[\Gamma_0]_l \ni 0$, $X \ni \Gamma_0$, or into $\hat{P}_0 \ni 0$, $X \ni \hat{P}_0$. This means:
1) in the first case, X decomposes into $X = \{\Gamma_0\} \cup \{X_1, \dots, X_K\}$ [2] with $[\Gamma_0]_l \ni 0$,

[1] We thank Daniel Ueltschi for pointing out this method of demonstration.

[2] The chains X_i are obtained as follows: consider the abstract connected graph $\hat{G}$ associated to the chain X. Then, remove all the edges of $\hat{G}$ that are adjacent to the vertex z_0 representing Γ_0 and z_0 itself, and consider the decomposition of the remaining graph into connected components. These components are exactly the representatives of $X_1, \dots, X_K$.

$d(X_i, \Gamma_0) \le R$, $n(X_i) \le N$ for all $i = 1, \dots, K$, $X_i \cap X_j = \emptyset$ for all $i \neq j$. The contribution to ξ_{N+1} is thus bounded by

$$\begin{aligned} &\sum_{\Gamma_0 : [\Gamma_0]_l \ni 0} \rho_0(\Gamma_0) e^{c|[\Gamma_0]_l|} \sum_{K \ge 0} \frac{1}{K!} \prod_{i=1}^{K} \sum_{\substack{X_i : d(X_i, \Gamma_0) \le R \\ n(X_i) \le N}} \omega_0(X_i) e^{c|b(X_i)|} \qquad (6.24) \\ &\le \sum_{\Gamma_0 : [\Gamma_0]_l \ni 0} \rho_0(\Gamma_0) e^{c|[\Gamma_0]_l|} \sum_{K \ge 0} \frac{1}{K!} \big(|[\Gamma_0]_R| \xi_N \big)^K \\ &\le \sum_{\Gamma_0 : [\Gamma_0]_l \ni 0} \rho_0(\Gamma_0) e^{(c+\epsilon)|[\Gamma_0]_l|} \le \frac{\epsilon}{2}, \end{aligned}$$

where we used the induction hypothesis $\xi_N \le \epsilon$.
2) in the second case, $X = \{\hat{P}_0\} \cup \{X_1, \dots, X_K\}$ with $\hat{P}_0 \ni 0$, $d(X_i, \hat{P}_0) \le R$, $n(X_i) \le N$ for all $i = 1, \dots, K$, $X_i \cap X_j = \emptyset$ for all $i \neq j$. A chain X_i of this decomposition can be of two types: i) there exists a cluster $\hat{P} \in X_i$ such that $\hat{P} \cap \hat{P}_0 \neq \emptyset$. Then the contribution from these chains is at most

$$|\hat{P}_0| \sum_{\substack{X_i : b(X_i) \ni 0 \\ n(X_i) \le N}} \omega_0(X_i) e^{c|b(X_i)|} = |\hat{P}_0| \xi_N \le |\hat{P}_0| \epsilon. \qquad (6.25)$$

ii) there exists $\Gamma \in X_i$, $\Gamma \cap \{[\hat{P}_0]_R\}_l \neq \emptyset$, where the thickening $\{\cdot\}_l$ was defined in (4.15). Notice that the set $\{[\hat{P}_0]_R\}_l \in \mathcal{C}^{(l)}$ contains at most $2^d |\hat{P}_0|$ cubes $C^{(l)}$. Since contours are composed of cubes $C^{(l)}$, the contribution from these chains can be bounded by

$$2^d |\hat{P}_0| \xi_N \le 2^d \epsilon |\hat{P}_0| \,. \qquad (6.26)$$

We can then proceed like in (6.24), and get a contribution to ξ_{N+1} bounded by

$$\sum_{\hat{P}_0 \ni 0} \lambda_0(\hat{P}_0) e^{c|\hat{P}_0|} e^{\epsilon(2^d+1)|\hat{P}_0|} \le \frac{\epsilon}{2} \,. \qquad (6.27)$$

Altogether, this shows that $\xi_{N+1} \le \epsilon$. □

6.3 Domains of Analyticity

In this section we consider the dependence of the weights $\omega^+(X)$ on the magnetic field $h \in \mathbb{C}$, in a neighbourhood of $\{\operatorname{Re} h = 0\}$. For obvious reasons, the domain in which $\omega^+(X)$ can be shown to be analytic depends on the contour $\Gamma \in X$ that has the largest interior. For the sake of simplicity, statements will be given only for chains of type $+$.

The domains of analyticity depend on the isoperimetric constants $K(N)$ defined in (4.56). Consider the reals

$$R(N) := \frac{\theta}{2K(N)N^{\frac{1}{d}}}\,, \tag{6.28}$$

where $\theta \in (0,1)$ will play an important role later in the study of the derivatives. We know from Lemma 4.10 that $R(N)N^{\frac{1}{d}}$ is increasing and that

$$\lim_{N\to\infty} R(N)N^{\frac{1}{d}} = \frac{\theta}{2K(\infty)}\,. \tag{6.29}$$

Since we want the domains of analyticity to be decreasing with the size of the contours, we define

$$R^*(N) := \min\{R(N') : 1 \leq N' \leq N\}\,. \tag{6.30}$$

The sequences $R^*(N)$ and $R(N)$ have the same asymptotic behaviour, as the following lemma shows.

Lemma 6.2.

$$\lim_{N\to\infty} R^*(N)N^{\frac{1}{d}} = \frac{\theta}{2K(\infty)}\,. \tag{6.31}$$

Proof. First notice that there exists an unbounded increasing sequence $N_1, N_2, \ldots$, such that $R^*(N_i) = R(N_i)$. This is a direct consequence of the bounds

$$R^*(N) \leq R(N) \leq \frac{\theta}{2K(\infty)N^{\frac{1}{d}}}\,. \tag{6.32}$$

Since $R(N)N^{\frac{1}{d}}$ increases, it is sufficient to show that $R^*(N)N^{\frac{1}{d}}$ is increasing. Consider the interval $[N, N+1]$. We have two possibilities: 1) $R(N+1) \geq R^*(N)$. In this case, $R^*(N+1)(N+1)^{\frac{1}{d}} = R^*(N)(N+1)^{\frac{1}{d}} \geq R^*(N)N^{\frac{1}{d}}$. 2) $R(N+1) \leq R^*(N)$. In this case, $R^*(N+1)(N+1)^{\frac{1}{d}} = R(N+1)(N+1)^{\frac{1}{d}} \geq R(N)N^{\frac{1}{d}} \geq R^*(N)N^{\frac{1}{d}}$. □

For $r > 0$, consider the strip

$$U(r) := \{z \in \mathbb{C} : |\mathrm{Re}\, z| < r\}\,. \tag{6.33}$$

Generally, we will restrict our attention to small magnetic fields, that is $h \in U_0 := U(h_0)$ where h_0 will be taken small enough. For instance, $h_0 < \frac{1}{16}$ so that the results on the restricted phases can be used in U_0.

We define the domain of analyticity for a contour:

$$U_\Gamma := U(R^*(V(\Gamma))) \cap U_0 \,, \tag{6.34}$$

and for a chain X:

$$U_X := \bigcap_{\Gamma \in X} U_\Gamma \,. \tag{6.35}$$

That is, $U_X = U_{\Gamma^{\max}}$, where $\Gamma^{\max} \in X$ has the largest interior. Notice that the domains U_Γ, U_X depend on θ. Set $V(X) := V(\Gamma^{\max}) = \max\{V(\Gamma) : \Gamma \in X\}$. The main result of this section is the following.

Proposition 6.1. *Let $\theta \in (0,1)$, $\epsilon > 0$, $c > 0$ small enough. There exists $\beta_1 = \beta_1(\theta,\epsilon)$ such that for all $\beta \geq \beta_1$, the following holds. For each chain X, $h \mapsto \omega^+(X)$ is analytic in U_X. Moreover,*

$$\|\omega^+(X)\|_{U_X} < \omega_0(X)\,, \quad \Big\|\frac{\mathrm{d}}{\mathrm{d}h}\omega^+(X)\Big\|_{U_X} < \omega_0(X)\,, \tag{6.36}$$

where $\omega_0(X)$ is defined via the weights $\lambda_0(\hat{P})$ and $\rho_0(\Gamma)$ given in (6.38)-(6.39) *hereafter, and satisfies* (6.22).

Before starting the proof of Proposition 6.1, we give explicitly the weights $\lambda_0(\hat{P})$ and $\rho_0(\Gamma)$. These weights are defined such that they can be used throughout the section, also when bounding the first derivative of $\omega^+(X)$. As will be seen, the non-trivial part of $\omega^+(\Gamma)$ will be bounded by:

$$\Big\|\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\Big\|_{U_\Gamma} \leq e^{\beta\theta\|\Gamma\|} e^{\frac{2}{3}|\Gamma|} \,. \tag{6.37}$$

Using (6.13) and (6.11), $\|\rho_1(\Gamma)\|_{U_0} \leq e^{-\beta\|\Gamma\|} e^{2\beta h_0 |\Gamma|} e^{\frac{1}{3}|\Gamma|}$. This suggests defining the weight $\rho_0(\Gamma)$ in the following way:

$$\rho_0(\Gamma) := D_1 \beta |\Gamma|^{\frac{d}{d-1}} e^{-(1-\theta)\beta\|\Gamma\|} e^{2\beta h_0|\Gamma|} e^{|\Gamma|} \,. \tag{6.38}$$

The term $D_1\beta|\Gamma|^{\frac{d}{d-1}}$ has been added to take into account other contributions, especially when studying the first derivative. For clusters we get, using the definition of $\lambda^+(\hat{P})$ and (5.54),

$$\begin{aligned} \|\lambda^+(\hat{P})\|_{H_+} &\leq \|\omega^+(\hat{P})\|_{H_+} e^{\|\omega^+(\hat{P})\|_{H_+}} \\ &\leq |\omega_0(\hat{P})| e^{|\omega_0(\hat{P})|} \leq |\omega_0(\hat{P})| e^{\epsilon_r} < D_2 |\omega_0(\hat{P})| \equiv \lambda_0(\hat{P}) \,. \end{aligned} \tag{6.39}$$

The numerical constants D_1, D_2 are assumed to be fixed and sufficiently large, in order to cover all the cases that will appear in the sequel.

Lemma 6.3. *Let $\theta \in (0,1)$, $c > 0$, and $\epsilon > 0$ be small enough. Assume $2h_0 \leq \frac{1}{2}(1-\theta)\rho$ (ρ is the Peierls constant). There exists $\beta_1 = \beta_1(\theta, \epsilon)$ such that for all $\beta \geq \beta_1$, the hypothesis* (6.21) *of Lemma 6.1 are satisfied.*

Proof. Define a new weight for polymers (see (5.46)):

$$\widetilde{\omega}_0(P) := \omega_0(P) e^{(c+\epsilon(2^d+1))|P|} \,. \tag{6.40}$$

If β is large enough, we can proceed as in (5.54) and get

$$\begin{aligned} \sum_{\hat{P} \ni 0} \lambda_0(\hat{P}) e^{(c+\epsilon(2^d+1))|\hat{P}|} &= D_2 \sum_{\hat{P} \ni 0} |\omega_0(\hat{P})| e^{(c+\epsilon(2^d+1))|\hat{P}|} \\ &\leq D_2 \sum_{\hat{P} \ni 0} |\widetilde{\omega}_0(\hat{P})| \leq \frac{\epsilon}{2} \,. \end{aligned} \tag{6.41}$$

This shows the first inequality of (6.21). For the second, we use the Peierls condition $\|\Gamma\| \geq \rho|\Gamma|$ (Proposition 4.2). This gives

$$\begin{aligned} \sum_{\Gamma : [\Gamma]_l \ni 0} \rho_0(\Gamma) e^{(c+\epsilon)|[\Gamma]_l|} &\leq D_1 \beta \sum_{\Gamma : [\Gamma]_l \ni 0} |\Gamma|^{\frac{d}{d-1}} e^{-(1-\theta)\beta\rho|\Gamma|} e^{2\beta h_0 |\Gamma|} e^{|\Gamma|} e^{(c+\epsilon)|[\Gamma]_l|} \\ &\leq D_1 \beta \sum_{\Gamma : [\Gamma]_l \ni 0} |\Gamma|^{\frac{d}{d-1}} e^{-\frac{1}{2}(1-\theta)\beta\rho|\Gamma|} e^{|\Gamma|} e^{(c+\epsilon)|[\Gamma]_l|} \,. \end{aligned}$$

Since $|[\Gamma]_l| \leq 3^d |\Gamma|$, a standard Peierls estimate allows to bound this sum by $\frac{\epsilon}{2}$ as soon as β is large enough. □

Until now we have denoted by $\epsilon_r = \epsilon_r(\beta)$ the small function appearing in the study of the restricted phases. Similarly, we denote by $\epsilon_c = \epsilon_c(\beta)$ the small function appearing in the study of chains. These two parameters are assumed to have a common bound $\max\{\epsilon_r, \epsilon_c\} \leq \epsilon$, which is small.

Consider the weight $\omega^+(\Gamma)$ given (6.12). We can use the linking procedure for the partition functions $\Theta^\pm(\text{int}\Gamma; \mp\sigma_\Gamma)$, yielding

$$\omega^+(\Gamma) = \rho_1(\Gamma) \frac{e^{-\beta h V(\Gamma)} \mathcal{Z}_r(\mathcal{P}^-_{\text{int}\Gamma}(+\sigma_\Gamma)) \Xi(\mathcal{X}^-_{\text{int}\Gamma}(+\sigma_\Gamma))}{e^{+\beta h V(\Gamma)} \mathcal{Z}_r(\mathcal{P}^+_{\text{int}\Gamma}(-\sigma_\Gamma)) \Xi(\mathcal{X}^+_{\text{int}\Gamma}(-\sigma_\Gamma))} \,. \tag{6.42}$$

Proof of Proposition 6.1: The proof will be done by induction, in the same spirit as in [Pf]. We say a contour Γ is of class n if $V(\Gamma) = n$. A chain is of class n if $V(X) = n$.

Consider a contour Γ of small class (say, of class smaller than l^d). Then the last ratio appearing in (6.42) equals 1. We bound $\omega^+(\Gamma)$ at $h = x + iy \in U_\Gamma$. First,

$$|e^{-2\beta h V(\Gamma)}| \leq e^{2\beta|x|V(\Gamma)} \leq e^{2\beta R^*(V(\Gamma))V(\Gamma)} \leq e^{2\beta R(V(\Gamma))V(\Gamma)} \leq e^{\theta\beta\|\Gamma\|} \,, \tag{6.43}$$

where we used the definition of the isoperimetric constants $K(\cdot)$ given in (4.56). Then, write

$$\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h} = \frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}} \frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}} \frac{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h} \tag{6.44}$$

The middle term has modulus 1 by symmetry (see (5.32)). The two other terms can be treated as follows:

$$\Big|\log \frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}\Big| = \Big|\int_0^x \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} \log \mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{s+iy}\Big| \leq |x|\epsilon_r V(\Gamma)\,. \tag{6.45}$$

We used Theorem 5.1. Proceeding as in (6.43), we get

$$\Big\|\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\Big\|_{U_\Gamma} \leq e^{\theta\epsilon_r\|\Gamma\|} \leq e^{\frac{1}{3}|\Gamma|}\,, \tag{6.46}$$

when β is large enough. Altogether this gives

$$\|\omega^+(\Gamma)\|_{U_\Gamma} \leq \|\rho_1(\Gamma)\|_{U_\Gamma} e^{\theta\beta\|\Gamma\|} e^{\frac{1}{3}|\Gamma|} \leq e^{-(1-\theta)\beta\|\Gamma\|} e^{2\beta h_0|\Gamma|} e^{2\frac{1}{3}|\Gamma|} < \rho_0(\Gamma)\,. \tag{6.47}$$

Since $\|\lambda^+(\hat{P})\|_{U_0} < \lambda_0(\hat{P})$, we have shown the first inequality of (6.36) for chains of small class. For the derivative, a Cauchy estimate (any disc centered at $h \in U_0$ with radius $\frac{1}{16}$ is contained in H_+) gives

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h}\lambda^+(\hat{P})\Big\|_{U_0} \leq 16\|\lambda^+(\hat{P})\|_{H_+}\,. \tag{6.48}$$

For contours,

$$\frac{\mathrm{d}}{\mathrm{d}h}\omega^+(\Gamma) = \omega^+(\Gamma)\frac{\mathrm{d}}{\mathrm{d}h}\log\omega^+(\Gamma) =$$
$$\omega^+(\Gamma)\Big(-\beta\frac{\mathrm{d}}{\mathrm{d}h}H_\Gamma(\sigma[\Gamma]) - \beta|\Gamma| + \frac{\mathrm{d}}{\mathrm{d}h}E^+_\Gamma - 2\beta V(\Gamma) + \frac{\mathrm{d}}{\mathrm{d}h}\log\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\Big)$$

Using $V(\Gamma) \leq |\Gamma|^{\frac{d}{d-1}}$ (this is a consequence of Lemma 4.9) and

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h}\log\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\Big\|_{U_\Gamma} \leq 2\epsilon_r V(\Gamma)\,, \tag{6.49}$$

this gives the upper bound

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h}\omega^+(\Gamma)\Big\|_{U_\Gamma} \leq 6\beta|\Gamma|^{\frac{d}{d-1}}\|\omega^+(\Gamma)\|_{U_\Gamma}\,, \tag{6.50}$$

which implies, as can be seen easily, that

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} \omega^+(X) \Big\|_{U_X} < \omega_0(X) \,. \tag{6.51}$$

With Lemma 6.1, this shows the proposition for chains of small class. Suppose it has been shown for chains of class $\leq n$. By this induction hypothesis, (6.22) and Lemma C.1, a cluster expansion can be used for the partition functions containing chains. Let X be a chain of class $n+1$, and consider $\Gamma \in X$. The treatment of the restricted phases is the same, and we must study the ratio

$$\frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h} = \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}} \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}} \frac{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_{iy}}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))_h} \,. \tag{6.52}$$

Again the middle term has modulus 1 and the rest is treated using the induction hypothesis.

$$\Big| \log \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_h}{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{iy}} \Big| = \Big| \int_0^x \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} \log \Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))_{s+iy} \Big| \leq |x| \epsilon_c V(\Gamma) \,. \tag{6.53}$$

This implies

$$\Big\| \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))} \Big\|_{U_\Gamma} \leq e^{\theta \epsilon_c \|\Gamma\|} \leq e^{\frac{1}{3}|\Gamma|} \,. \tag{6.54}$$

For the weight of Γ, we thus have (compare with (6.47)):

$$\|\omega^+(\Gamma)\|_{U_\Gamma} \leq e^{-(1-\theta)\beta\|\Gamma\|} e^{2\beta h_0 |\Gamma|} e^{3\frac{1}{3}|\Gamma|} < \rho_0(\Gamma) \,. \tag{6.55}$$

For the derivative, use again the induction hypothesis, and bound

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} \log \frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))} \Big\|_{U_\Gamma} \leq 2\epsilon_c V(\Gamma) \,. \tag{6.56}$$

It is easy to check that (6.50) still holds which, in turn, implies (6.51). This shows the proposition. □

6.4 Pure Phases

In the last section we gave for each chain X a domain U_X in which the weight $\omega^+(X)$ behaves analytically. The size of the domain U_X shrinks to $\{\mathrm{Re}\, h = 0\}$ when the size of the largest contour of X increases. In the present section we show that the weights $\omega^+(X)$ can actually be controlled when $0 < \mathrm{Re}\, h < h_+$ where h_+ is fixed, independently of the size of X.

We consider only chains of type +, the case − being similar by symmetry. Define

$$U_+ := \{z \in \mathbb{C} : 0 < \mathrm{Re}\, h < h_+\}\,, \tag{6.57}$$

where $0 < h_+ \leq \min\{\frac{1}{16}, \frac{\rho}{2}\}$ is fixed (ρ is the Peierls constant). In Chapter 7, domains will have to be made optimal, with θ close to 1, but here we choose $\theta := \frac{1}{2}$. The main result of this section is the following

Proposition 6.2. *Let $\epsilon, c > 0$ be small enough. There exists $\beta_2 = \beta_2(\epsilon)$ such that for all $\beta \geq \beta_2$, the following holds. For each chain X of type +, $h \mapsto \omega^+(X)$ is analytic in U_+, and*

$$\|\omega^+(X)\|_{U_+} \leq \omega_0(X)\,, \tag{6.58}$$

where $\omega_0(X)$ satisfies (6.22).

Proof. Since $U_+ \subset H_+$, clusters $\hat{P}$ and restricted phases are under control. For each Γ, we use the representation (6.12) (rather than (6.42)). The main ingredient of the proof is the following lemma, whose proof is standard (see [Z] or Appendix A, with minor modifications due to the fact that we are working with analytic restricted phases rather than ground states).

Lemma 6.4. *Let β be large enough. Then for each contour Γ of type +, we have $\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma) \neq 0$ on U_+ and*

$$\Big\|\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\Big\|_{U_+} \leq e^{\frac{2}{3}|\Gamma|}\,. \tag{6.59}$$

Proof. For contours of small enough class, we can proceed as in (6.44)-(6.45):

$$\Big\|\frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}\Big\|_{U_+} = \Big\|e^{-2\beta h V(\Gamma)}\frac{\mathscr{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma}(+\sigma_\Gamma))}{\mathscr{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma}(-\sigma_\Gamma))}\Big\|_{U_+} \tag{6.60}$$

$$\leq \sup_{h \in U_+} e^{-2\beta \mathrm{Re}\, h V(\Gamma)} e^{2\beta|\mathrm{Re}\, h| V(\Gamma)} \leq 1. \tag{6.61}$$

This shows (6.59). Suppose (6.59) has been shown for all contours of class $\leq n$. In particular, for each such contour,

$$\|\omega^+(\Gamma)\|_{U_+} \leq e^{-\beta\|\Gamma\|} e^{\beta h_+|\Gamma|} e^{|\Gamma|} < \rho_0(\Gamma)\,. \tag{6.62}$$

Consider a contour Γ of class $n+1$. For ease of notation we denote for a while $\Lambda \equiv \mathrm{int}\Gamma$. Until the end of the proof, we *fix* $\tilde{h} = x + iy \in U_+$, and show that

$$\Big|\frac{\Theta^-(\Lambda; +\sigma_\Gamma)}{\Theta^+(\Lambda; -\sigma_\Gamma)}\Big|_{\tilde{h}} \leq e^{\frac{2}{3}|\Gamma|}\,. \tag{6.63}$$

Notice that if $\tilde{h} \in U_\Gamma$ then (6.63) follows from (6.46) and (6.54), so we assume that $\tilde{h} \notin U_\Gamma$.

We start with $\Theta^+(\Lambda; -\sigma_\Gamma)$. Since (6.62) holds for all contour Γ of class $\le n$, we can apply the linking procedure:

$$\Theta^+(\Lambda; -\sigma_\Gamma) = e^{\beta h|\Lambda|} \mathcal{Z}_r(\mathcal{P}^+_\Lambda(-\sigma_\Gamma)) \Xi(\mathcal{X}^+_\Lambda(-\sigma_\Gamma)) \,. \tag{6.64}$$

We expand the partition function $\mathcal{Z}_r(\mathcal{P}^+_\Lambda(-\sigma_\Gamma))$, and use (5.57). The same can be done for the partition function containing chains:

$$\Xi(\mathcal{X}^+_\Lambda(-\sigma_\Gamma)) = e^{\beta \mathfrak{g}^+_n |\Lambda|} e^{\triangle^+_c(\Lambda)} \,, \tag{6.65}$$

where $\mathfrak{g}^+_n$ is defined by

$$\mathfrak{g}^+_n := \lim_{M\to\infty} \frac{1}{\beta|\Lambda_M|} \log \Xi^+_n(\Lambda_M) \,, \tag{6.66}$$

and $\Xi^+_n(\Lambda_M)$ is restricted to contain only chains in which each contour is of class at most n. We thus have

$$\Theta^+(\Lambda; -\sigma_\Gamma) = \left(e^{\beta \mathfrak{p}^+_{r,\gamma}|\Lambda|} e^{\triangle^+_r(\Lambda)}\right)\left(e^{\beta \mathfrak{g}^+_n|\Lambda|} e^{\triangle^+_c(\Lambda)}\right) \equiv e^{\beta \mathfrak{p}^+_n |\Lambda|} e^{\triangle^+(\Lambda)} \,, \tag{6.67}$$

where $\mathfrak{p}^+_n := \mathfrak{p}^+_{r,\gamma} + \mathfrak{g}^+_n$. The function $\triangle^+(\Lambda)$ depends on $-\sigma_\Gamma$ but satisfies $|\triangle^+(\Lambda)| \le \varepsilon_0 |\Gamma|$ with $\varepsilon_0 \le \frac{2}{9}$ when β is large enough.

We turn to $\Theta^-(\Lambda; +\sigma_\Gamma)$. An external contour Γ' appearing in $\Theta^-(\Lambda; +\sigma_\Gamma)$ is called stable if $U_{\Gamma'} \ni \tilde{h}$, and unstable if $U_{\Gamma'} \not\ni \tilde{h}$. Following Zahradník, we re-sum over external stable contours, yielding

$$\Theta^-(\Lambda; +\sigma_\Gamma) = \sum_{\substack{\{\Gamma'\}\subset\Lambda \\ \text{ext., unst.}}} \Theta^-_s(\text{ext}; \sigma_\Gamma \sigma_{\{\Gamma'\}}) \prod_{\Gamma'} \rho(\Gamma') \Theta^+(\text{int}\Gamma'; \sigma_{\Gamma'}) \,, \tag{6.68}$$

where $\text{ext} = \Lambda \cap \bigcap_{\Gamma'} \text{ext}\Gamma'$, and $\Theta^-_s(\text{ext}; \sigma_\Gamma \sigma_{\{\Gamma'\}})$ contains only stable contours. Applying the linking procedure and expanding, we have

$$\Theta^-_s(\text{ext}; \sigma_\Gamma \sigma_{\{\Gamma'\}}) = \left(e^{\beta \mathfrak{p}^-_{r,\gamma}|\text{ext}|} e^{\triangle^-_r(\text{ext})}\right)\left(e^{\beta \mathfrak{g}^-_s|\text{ext}|} e^{\triangle^-_c(\text{ext})}\right) \tag{6.69}$$

$$\equiv e^{\beta \mathfrak{p}^-_s |\text{ext}|} e^{\triangle^-(\text{ext})} \,, \tag{6.70}$$

where $\mathfrak{p}^-_s := \mathfrak{p}^-_{r,\gamma} + \mathfrak{g}^-_s$, and $|\triangle^-(\text{ext})| \le \varepsilon_0|\Gamma| + \varepsilon_0 \sum_{\Gamma'} |\Gamma'|$. The function $\mathfrak{g}^-_s$ is defined like in (6.66), containing only stable contours (of type $-$). The partition functions $\Theta^+(\text{int}\Gamma'; \sigma_{\Gamma'})$ can be treated like $\Theta^+(\Lambda; -\sigma_\Gamma)$, yielding

$$\Theta^+(\text{int}\Gamma'; \sigma_{\Gamma'}) = e^{\beta \mathfrak{p}^+_n |\text{int}\Gamma'|} e^{\triangle^+(\text{int}\Gamma')} \,, \tag{6.71}$$

with $|\triangle^+(\mathrm{int}\Gamma')| \leq \varepsilon_0|\Gamma'|$. Altogether, the ratio (6.63) equals

$$\Big|e^{-\triangle^+(\Lambda)} \sum_{\substack{\{\Gamma'\}\subset\Lambda\\ \text{ext., unst.}}} e^{\beta(\mathfrak{p}_s^- - \mathfrak{p}_n^+)|\mathrm{ext}|} e^{\triangle^-(\mathrm{ext})} \prod_{\Gamma'} \rho(\Gamma') e^{-\beta\mathfrak{p}_n^+|\Gamma|} e^{\triangle^+(\mathrm{int}\Gamma')}\Big|_{\tilde{h}} \tag{6.72}$$

$$\leq e^{2\varepsilon_0|\Gamma|} \sum_{\substack{\{\Gamma'\}\subset\Lambda\\ \text{ext., unst.}}} e^{\beta\mathrm{Re}\,(\mathfrak{p}_s^- - \mathfrak{p}_n^+)|\mathrm{ext}|} \prod_{\Gamma'} |\rho(\Gamma')| e^{-\beta\mathrm{Re}\,\mathfrak{p}_n^+|\Gamma|} e^{2\varepsilon_0|\Gamma'|}\,. \tag{6.73}$$

If we use the bounds $|\mathfrak{g}_n^+| \leq \varepsilon_0$, $h_+ \leq \frac{\varrho}{2}$ we get

$$\Big|\frac{\Theta^-(\Lambda;+\sigma_\Gamma)}{\Theta^+(\Lambda;-\sigma_\Gamma)}\Big|_{\tilde{h}} \leq e^{2\varepsilon_0|\Gamma|} \sum_{\substack{\{\Gamma'\}\subset\Lambda\\ \text{ext., unst.}}} e^{\beta\mathrm{Re}\,(\mathfrak{p}_s^- - \mathfrak{p}_n^+)|\mathrm{ext}|} \prod_{\Gamma'} e^{-\beta(\rho'+2\varepsilon_0)|\Gamma'|}\,, \tag{6.74}$$

where $\rho' := \frac{\varrho}{2} - 6\varepsilon_0$. Our aim is to show that $\mathrm{Re}\,(\mathfrak{p}_s^- - \mathfrak{p}_n^+) \leq -\mathfrak{e}_{\rho'}$, where $\mathfrak{e}_{\rho'}$ is the pressure of an auxiliary model.

An auxiliary contour model. Consider the weight $\eta(\cdot)$ defined by $\eta(\Gamma') = e^{-\beta\rho'|\Gamma'|}$ if Γ' is unstable, 0 otherwise. Define its associated partition function

$$\Xi'(\Lambda) := \sum_{\{\Gamma'\}\subset\Lambda} \prod_{\Gamma'} \eta(\Gamma')\,, \tag{6.75}$$

where the sum is over all families of contours $\{\Gamma'_1,\ldots,\Gamma'_k\}$ of type +, such that $d(\Gamma'_i,\Gamma'_j) > l$ for all $i \neq j$, $d(\Gamma'_i,\Lambda^c) > l$ for all i. Consider the associated function

$$\mathfrak{e}_{\rho'} := \lim_{M\to\infty} \frac{1}{\beta|\Lambda_M|} \log \Xi'(\Lambda_M)\,. \tag{6.76}$$

Remember that Γ' is unstable if and only if $U_{\Gamma'} \not\ni \tilde{h} = x+iy$, i.e. if $R^*(V(\Gamma')) < x$. This implies

$$|\Gamma'| > V(\Gamma')^{\frac{d-1}{d}} \geq \Big(\frac{1}{4K(1)x}\Big)^d \equiv L(x)\,. \tag{6.77}$$

We thus have a constant C_1 such that

$$|\mathfrak{e}_{\rho'}| \leq C_1 e^{-\beta\rho' L(x)} < \varepsilon_0\,. \tag{6.78}$$

Moreover for any Λ we have

$$\Xi'(\Lambda) = e^{\beta\mathfrak{e}_{\rho'}|\Lambda|} e^{\triangle_{\rho'}(\Lambda)}\,, \tag{6.79}$$

with $|\triangle_{\rho'}(\Lambda)| \leq \varepsilon_0|\partial_l^-\Lambda|$.

Study of the difference $\mathfrak{p}_s^- - \mathfrak{p}_n^+$. In the same way as we defined $\mathfrak{g}_s^-$, we define $\mathfrak{g}_s^+$, which is constructed with stable contours, of type $+$. Let $\mathfrak{p}_s^+ := \mathfrak{p}_{r,\gamma}^+ + \mathfrak{g}_s^+$, and write

$$\begin{aligned} \mathfrak{p}_s^- - \mathfrak{p}_n^+ &= (\mathfrak{p}_s^- - \mathfrak{p}_s^+) + (\mathfrak{p}_s^+ - \mathfrak{p}_n^+) \\ &= (\mathfrak{p}_{r,\gamma}^- - \mathfrak{p}_{r,\gamma}^+) + (\mathfrak{g}_s^- - \mathfrak{g}_s^+) + (\mathfrak{g}_s^+ - \mathfrak{g}_n^+)\,. \end{aligned} \tag{6.80}$$

Notice that the definition of the functions $\mathfrak{g}_s^\pm$ depends on $\tilde{h}$, but the maps $h \mapsto \mathfrak{g}_s^\pm(h)$ are analytic in the strip $\{h \in \mathbb{C} : |\mathrm{Re}\, h| < \mathrm{Re}\, \tilde{h}\}$. Using Proposition 6.1, we have, uniformly on this strip,

$$\Big|\frac{\mathrm{d}}{\mathrm{d}h}\mathfrak{g}_s^\pm\Big| \leq \epsilon_c\,. \tag{6.81}$$

Moreover, symmetry implies $\overline{\mathfrak{g}_s^+(iy)} = \mathfrak{g}_s^-(iy)$. This gives

$$\begin{aligned} |\mathrm{Re}\,(\mathfrak{g}_s^- - \mathfrak{g}_s^+)(\tilde{h})| &= |\mathrm{Re}\,(\mathfrak{g}_s^-(\tilde{h}) - \mathfrak{g}_s^-(iy)) + \mathrm{Re}\,(\mathfrak{g}_s^+(iy) - \mathfrak{g}_s^+(\tilde{h}))| \\ &= \Big| \int_0^x \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Re}\,\mathfrak{g}_s^-(s+iy)\mathrm{d}s + \int_x^0 \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Re}\,\mathfrak{g}_s^+(s+iy)\mathrm{d}s\Big| \leq 2\epsilon_c x \end{aligned} \tag{6.82}$$

The same can be done for the restricted pressures. Extracting from each $\mathfrak{p}_{r,\gamma}^\pm$ the linear term $\pm\tilde{h}$, we get

$$\mathrm{Re}\,(\mathfrak{p}_{r,\gamma}^- - \mathfrak{p}_{r,\gamma}^+)(\tilde{h}) \leq -2x + 2\epsilon_r x\,. \tag{6.83}$$

Consider the difference $\mathfrak{g}_s^+ - \mathfrak{g}_n^+$. Each cluster of chains contributing to this difference contains at least one chain X such that there exists an unstable contour $\Gamma \in X$. By (6.77) this implies the existence of two constants C_2, C_3, such that

$$|\mathfrak{g}_s^+ - \mathfrak{g}_n^+| \leq C_2 e^{-\beta C_3 L(x)}\,. \tag{6.84}$$

Altogether we get (once β is large enough):

$$\mathrm{Re}\,(\mathfrak{p}_s^- - \mathfrak{p}_n^+) + \mathfrak{e}_{\rho'} \leq -2x + 2\epsilon_r x + 2\epsilon_c x + C_2 e^{-\beta C_3 L(x)} + C_1 e^{-\beta\rho' L(x)} < 0\,.$$

The final step is then to bound (6.74) by

$$\begin{aligned} &e^{2\varepsilon_0|\Gamma|}e^{-\beta\mathfrak{e}_{\rho'}|\Lambda|} \sum_{\substack{\{\Gamma'\}\subset\Lambda \\ \text{ext., unst.}}} \prod_{\Gamma'} e^{-\beta(\rho'+2\varepsilon_0)|\Gamma'|} e^{+\beta\mathfrak{e}_{\rho'}(|\Gamma'|+V(\Gamma'))} \\ &\leq e^{2\varepsilon_0|\Gamma|}e^{-\beta\mathfrak{e}_{\rho'}|\Lambda|} \sum_{\substack{\{\Gamma'\}\subset\Lambda \\ \text{ext., unst.}}} \prod_{\Gamma'} e^{-\beta\rho'|\Gamma'|}\Xi'(\mathrm{int}\Gamma') \\ &= e^{2\varepsilon_0|\Gamma|}e^{-\beta\mathfrak{e}_{\rho'}|\Lambda|}\Xi'(\Lambda) \leq e^{2\varepsilon_0|\Gamma|}e^{-\beta\mathfrak{e}_{\rho'}|\Lambda|}e^{+\beta\mathfrak{e}_{\rho'}|\Lambda|}e^{\triangle_{\rho'}(\Lambda)} \leq e^{3\varepsilon_0|\Gamma|} \leq e^{\frac{2}{3}|\Gamma|}\,. \end{aligned} \tag{6.85}$$

In (6.85), we used $e^{\beta\mathfrak{e}_{\rho'}V(\Gamma')} \leq \Xi'(\mathrm{int}\Gamma')e^{\varepsilon_0|\Gamma'|}$. This shows (6.63). The same can be done for any $\tilde{h} \in U_+$, yielding (6.59) for contours of class $n+1$. □

The proof of Proposition 6.2 finishes by using Lemma 6.1. □

Chapter 7

Derivatives of the Pressure

In this section we adapt the mechanism used by Isakov to obtain lower bounds on the derivatives of the pressure (which was briefly presented in the Introduction), and prove Theorem 3.4. Although estimates of Theorem 3.4 were given for the pressure *density* $\mathfrak{p}_\gamma$, we will always work in a finite volume Λ, and obtain bounds on the derivatives that are uniform in the volume. Like in the preceding section, we assume $\gamma \in (0, \gamma_0)$ is fixed.

We consider a box $\Lambda = [-M, +M]^d \cap \mathbf{Z}^d$, with M large, chosen so that $\Lambda \in \mathcal{C}^{(l)}$. Outside Λ we fix the spins to the value $+1$, i.e. we consider the set Ω^+_Λ, defined in (6.2) and the associated partition function $Z^+(\Lambda)$ defined in (6.1). The finite volume pressure $\mathfrak{p}^+_{\gamma,\Lambda}$ is defined by

$$\mathfrak{p}^+_{\gamma,\Lambda} := \frac{1}{\beta|\Lambda|} \log Z^+(\Lambda)\,. \tag{7.1}$$

Clearly, this function equals the density pressure of (3.5) in the thermodynamic limit. Consider the set $\mathcal{C}^+(\Lambda)$ of *all possible* external contours of type + associated to the set Ω^+_Λ. Remember that $V(\Gamma) = |\mathrm{int}\Gamma|$, where $\mathrm{int}\Gamma$ denotes the union of all components of Γ^c with label $-$. The family $\mathcal{C}^+(\Lambda)$ can be totally ordered, with an order relation denoted $\preceq$, such that $V(\Gamma') \leq V(\Gamma)$ when $\Gamma' \preceq \Gamma$. When Γ is not the smallest contour we denote its predecessor (w.r.t. $\preceq$) by $i(\Gamma)$.

For a given external contour $\Gamma \in \mathcal{C}^+(\Lambda)$, consider the set

$$\Omega^+_\Lambda(\Gamma) := \{\sigma_\Lambda \in \Omega^+_\Lambda : \Gamma' \preceq \Gamma \text{ for all external contour } \Gamma' \text{ of } \sigma_\Lambda +_{\Lambda^c}\}\,,$$

and define the partition function

$$\Theta^+_\Gamma(\Lambda) := \sum_{\sigma_\Lambda \in \Omega^+_\Lambda(\Gamma)} \exp\big(-\beta H_\Lambda(\sigma_\Lambda +_{\Lambda^c})\big)\,. \tag{7.2}$$

When Γ is the largest contour then clearly $\Theta^+_\Gamma(\Lambda) = Z^+(\Lambda)$ and when Γ is the smallest contour, we define $\Theta^+_{i(\Gamma)}(\Lambda) := Z^+_r(\Lambda)$. We also introduce the following set in which the presence of Γ is *forced*:

$$\Omega^+_\Lambda[\Gamma] := \{\sigma_\Lambda \in \Omega^+_\Lambda : \Gamma' \preceq \Gamma \text{ for all external contour } \Gamma' \text{ of } \sigma_\Lambda +_{\Lambda^c}, \\ \text{and } \Gamma \text{ is a contour of } \sigma_\Lambda +_{\Lambda^c}\} . \tag{7.3}$$

We have $\Omega^+_\Lambda[\Gamma] \subset \Omega^+_\Lambda(\Gamma)$. The partition function $\Theta^+_{[\Gamma]}(\Lambda)$ is defined as (7.2), with $\Omega^+_\Lambda[\Gamma]$ in place of $\Omega^+_\Lambda(\Gamma)$. We have the following fundamental identity:

$$\Theta^+_\Gamma(\Lambda) = \Theta^+_{i(\Gamma)}(\Lambda) + \Theta^+_{[\Gamma]}(\Lambda) . \tag{7.4}$$

A crucial idea of Isakov is to consider the following identity.

$$Z^+(\Lambda) = Z^+_r(\Lambda) \prod_{\Gamma \in \mathcal{C}^+(\Lambda)} \frac{\Theta^+_\Gamma(\Lambda)}{\Theta^+_{i(\Gamma)}(\Lambda)} . \tag{7.5}$$

Then, the logarithm is written as a *finite* sum:

$$\log Z^+(\Lambda) = \log Z^+_r(\Lambda) + \sum_{\Gamma \in \mathcal{C}^+(\Lambda)} u^+_\Lambda(\Gamma) , \tag{7.6}$$

where

$$u^+_\Lambda(\Gamma) := \log \frac{\Theta^+_\Gamma(\Lambda)}{\Theta^+_{i(\Gamma)}(\Lambda)} . \tag{7.7}$$

Using (7.4) we can write $u^+_\Lambda(\Gamma) = \log(1 + \varphi^+_\Lambda(\Gamma))$, where

$$\varphi^+_\Lambda(\Gamma) := \frac{\Theta^+_{[\Gamma]}(\Lambda)}{\Theta^+_{i(\Gamma)}(\Lambda)} . \tag{7.8}$$

Non-analyticity of the pressure is examined by studying high order derivatives of the functions $\varphi^+_\Lambda(\Gamma)$ at $h = 0$, using Cauchy's Formula

$$\varphi^+_\Lambda(\Gamma)^{(k)}(0) = \frac{k!}{2\pi i} \int_C \frac{\varphi^+_\Lambda(\Gamma)(z)}{z^{k+1}} \mathrm{d}z . \tag{7.9}$$

To obtain bounds on $\varphi^+_\Lambda(\Gamma)^{(k)}(0)$, we exponentiate $\varphi^+_\Lambda(\Gamma)$ and use a stationary phase analysis to estimate the integral. The contour C will be chosen in a k-dependent way. If the domain $U_\Gamma \ni 0$ in which $\varphi^+_\Lambda(\Gamma)$ is analytic is too small, then no information (not even the sign!) can be given about $\varphi^+_\Lambda(\Gamma)^{(k)}(0)$.

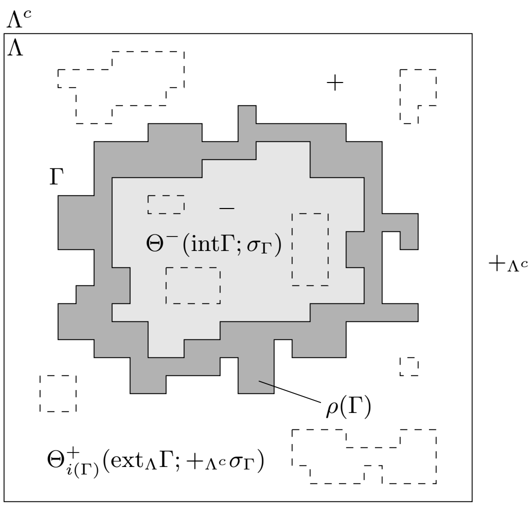

Figure 7.1: The decomposition (7.10) of the partition function $\Theta^+_{[\Gamma]}(\Lambda)$.

For a while, consider the structure of the partition function $\Theta^+_{[\Gamma]}(\Lambda)$. We write $\Lambda = \mathrm{ext}_\Lambda\Gamma \cup \Gamma \cup \mathrm{int}\Gamma$, where $\mathrm{ext}_\Lambda\Gamma := \mathrm{ext}\Gamma \cap \Lambda$. By construction, $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$ are at distance at least $l > 2R$. We will therefore consider $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$ as independent systems (see Figure 7.1). The sums over configurations on $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$ can be done separately, yielding

$$\Theta^+_{[\Gamma]}(\Lambda) = \rho(\Gamma)\Theta^+_{i(\Gamma)}(\mathrm{ext}_\Lambda\Gamma;+_{\Lambda^c}\sigma_\Gamma)\Theta^-(\mathrm{int}\Gamma;\sigma_\Gamma)\,. \tag{7.10}$$

All the contours of these partition functions are at distance larger than l from Γ, and have an interior smaller than $V(\Gamma)$. The point is that we control these functions for $h \in U_\Gamma$, where $U_\Gamma \subset \mathbb{C}$ is a domain that depends *only on the volume of* Γ.

The program for the rest of the section is the following. In Section 7.1 we show that $\varphi^+_\Lambda(\Gamma)$ can be exponentiated, using the results of Chapter 6. We then use a stationary phase analysis and obtain upper and lower bounds on some derivatives of $\varphi^+_\Lambda(\Gamma)$ and $u^+_\Lambda(\Gamma)$ at $h = 0$. In Section 7.2 we fix k and take the box Λ large enough. For a class of contours called k-large and thin, the k-th derivative of $u^+_\Lambda(\Gamma)$ can be estimated from below, using the results of Section 7.1. This gives a lower bound on $\mathfrak{p}^{+(k)}_{\gamma,\Lambda}(0)$. In Section 7.3 we show that for $\mathfrak{p}^+_{\gamma,\Lambda}$, the operations $\lim_\Lambda$ and $(\cdot)^{(k),\leftarrow}(0)$ commute, leading to the proof of our main results.

7.1 Study of the Functions $\varphi^+_\Lambda(\Gamma)$

The proof of the following lemma requires the main results of Chapter 5 and 6. After that, the proof of non-analyticity of the pressure will essentially follow the argument of Isakov (see [Isakov1], [Isakov2] or Appendix A).

Lemma 7.1. *Let $\theta \in (0,1)$, β large enough. Then the following holds. For all contour $\Gamma \in \mathcal{C}^+(\Lambda)$ with $V(\Gamma) \neq 0$ there exists a map $h \mapsto g^+_\Lambda(\Gamma)(h)$ analytic in the strip U_Γ, such that for all $h \in U_\Gamma$, $\varphi^+_\Lambda(\Gamma)$ can be exponentiated:*

$$\varphi^+_\Lambda(\Gamma) = \exp\big(- \beta\|\Gamma\| - 2\beta h V(\Gamma) + 2\beta V(\Gamma) g^+_\Lambda(\Gamma)\big) . \tag{7.11}$$

Moreover, we have the following local estimate

$$2\beta V(\Gamma)|g^+_\Lambda(\Gamma)(0)| \leq \delta_1(\beta)\beta\|\Gamma\| , \tag{7.12}$$

and a uniform bound on the first derivative

$$\Big\|\frac{\mathrm{d}}{\mathrm{d}h} g^+_\Lambda(\Gamma)\Big\|_{U_\Gamma} \leq \delta_2(\beta) + 2\frac{|\Gamma|}{V(\Gamma)} . \tag{7.13}$$

The functions δ_i are such that $\lim_{\beta\nearrow\infty} \delta_i = 0$.

Proof. Consider $\Theta^+_{[\Gamma]}(\Lambda)$. We have seen how to re-sum over configurations on $\mathrm{ext}_\Lambda\Gamma$ and $\mathrm{int}\Gamma$. We write

$$\varphi^+_\Lambda(\Gamma) = \rho(\Gamma) \frac{\Theta^+_{i(\Gamma)}(\mathrm{ext}_\Lambda\Gamma; +_{\Lambda^c}\sigma_\Gamma)\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)}{\Theta^+_{i(\Gamma)}(\Lambda)} \frac{\Theta^-(\mathrm{int}\Gamma; +\sigma_\Gamma)}{\Theta^+(\mathrm{int}\Gamma; -\sigma_\Gamma)} . \tag{7.14}$$

All the volume contributions coming from the first quotient will be shown to vanish. The partition functions $\Theta^+_{i(\Gamma)}(\mathrm{ext}_\Lambda\Gamma; +_{\Lambda^c}\sigma_\Gamma)$ and $\Theta^\pm(\mathrm{int}\Gamma; \mp\sigma_\Gamma)$ are of the type (6.5). We can therefore apply the linking procedure and obtain a representation of the form (6.19) for each of them:

$$\Theta^+_{i(\Gamma)}(\mathrm{ext}_\Lambda\Gamma; +_{\Lambda^c}\sigma_\Gamma) = e^{\beta h|\mathrm{ext}_\Lambda\Gamma|} \mathcal{Z}_r(\mathcal{P}^+_{\mathrm{ext}_\Lambda\Gamma}) \Xi(\mathcal{X}^+_{\mathrm{ext}_\Lambda\Gamma}) \tag{7.15}$$

$$\Theta^\pm(\mathrm{int}\Gamma; \mp\sigma_\Gamma) = e^{\pm\beta h V(\Gamma)} \mathcal{Z}_r(\mathcal{P}^\pm_{\mathrm{int}\Gamma}) \Xi(\mathcal{X}^\pm_{\mathrm{int}\Gamma}) , \tag{7.16}$$

where we omitted, in the notation, to mention that the families of polymers and chains always depend on the boundary conditions specified by $+_{\Lambda^c}$ and σ_Γ. Moreover, the family $\mathcal{X}^+_{\mathrm{ext}_\Lambda\Gamma}$ contains chains X that satisfy $V(X) \leq V(\Gamma)$. In the same way:

$$\Theta^+_{i(\Gamma)}(\Lambda) = e^{\beta h|\Lambda|} \mathcal{Z}_r(\mathcal{P}^+_\Lambda) \Xi(\mathcal{X}^+_\Lambda) , \tag{7.17}$$

where the families $\mathcal{P}^+_\Lambda$ and $\mathcal{X}^+_\Lambda$ depend only on the boundary condition $+_{\Lambda^c}$. Using the definition of $\rho(\Gamma)$, it is easy to see that $\varphi^+_\Lambda(\Gamma)$ has the form (7.11), where $g^+_\Lambda(\Gamma)$ is defined by

$$2\beta V(\Gamma) g^+_\Lambda(\Gamma) := -\beta \sum_{i\in\Gamma} u((\sigma_\Gamma)_i) - \beta h|\Gamma| + \log Q_r + \log Q_c , \tag{7.18}$$

where $u(\sigma_i) = -h\sigma_i$, and the quotients Q_r, Q_c are defined by

$$Q_r(h) := \frac{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{ext}_\Lambda\Gamma})\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^+_\Lambda)}\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma})}, \tag{7.19}$$

$$Q_c(h) := \frac{\Xi(\mathcal{X}^+_{\mathrm{ext}_\Lambda\Gamma})\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma})}{\Xi(\mathcal{X}^+_\Lambda)}\frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma})}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma})}. \tag{7.20}$$

Since all the families of chains involved contain contours with an interior smaller than Γ, $h \mapsto g_\Lambda^+(\Gamma)$ is analytic in the strip U_Γ (by Proposition 6.1). Rearranging the terms of the cluster expansions for Q_r leads to

$$\log Q_r = \log\frac{\mathcal{Z}_r(\mathcal{P}^-_{\mathrm{int}\Gamma})}{\mathcal{Z}_r(\mathcal{P}^+_{\mathrm{int}\Gamma})} + \sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+_{\mathrm{ext}_\Lambda\Gamma}\\ \hat{P}\cap[\Gamma]_R\neq\emptyset}}\omega^+(\hat{P}) + \sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+_{\mathrm{int}\Gamma}\\ \hat{P}\cap[\Gamma]_R\neq\emptyset}}\omega^+(\hat{P}) - \sum_{\substack{\hat{P}\in\hat{\mathcal{P}}^+_{\Lambda}\\ \hat{P}\cap[\Gamma]_R\neq\emptyset}}\omega^+(\hat{P}).$$

Notice that the volume contributions from $\mathrm{ext}_\Lambda\Gamma$ cancelled, and that the three sums are boundary terms. By symmetry, the quotient equals 1 at $h = 0$, and so

$$|\log Q_r(0)| \leq 3\epsilon_r|[\Gamma]_R|. \tag{7.21}$$

For the derivative, using (5.56) gives

$$\left\|\frac{\mathrm{d}}{\mathrm{d}h}\log Q_r\right\|_{\tilde{H}_+} \leq 2\epsilon_r V(\Gamma) + 3\epsilon_r|[\Gamma]_R|. \tag{7.22}$$

The same computations can be done for Q_c. Clusters of chains are denoted $\hat{X}$. The contributions from $\mathrm{ext}_\Lambda\Gamma$ also cancel. Indeed, consider the difference

$$\sum_{\hat{X}\in\hat{\mathcal{X}}^+_{\mathrm{ext}_\Lambda\Gamma}}\omega^+(\hat{X}) - \sum_{\hat{X}\in\hat{\mathcal{X}}^+_{\Lambda}}\omega^+(\hat{X}). \tag{7.23}$$

Using Lemma 4.5, there exists for all $\hat{X}_1 \in \hat{\mathcal{X}}^+_{\mathrm{ext}_\Lambda\Gamma}$ with $d(\hat{X}_1, \Gamma) > R$, a cluster $\hat{X}_2 \in \hat{\mathcal{X}}^+_\Lambda$, $\hat{X}_2 \cap \mathrm{ext}_\Lambda\Gamma \neq \emptyset$, $d(\hat{X}_2, \Gamma) > R$, such that $\omega^+(\hat{X}_1) = \omega^+(\hat{X}_2)$. We are thus left with

$$\log Q_c = \log\frac{\Xi(\mathcal{X}^-_{\mathrm{int}\Gamma})}{\Xi(\mathcal{X}^+_{\mathrm{int}\Gamma})} + \sum_{\substack{\hat{X}\in\hat{\mathcal{X}}^+_{\mathrm{ext}_\Lambda\Gamma}\\ \hat{X}\cap[\Gamma]_R\neq\emptyset}}\omega^+(\hat{X}) + \sum_{\substack{\hat{X}\in\hat{\mathcal{X}}^+_{\mathrm{int}\Gamma}\\ \hat{X}\cap[\Gamma]_R\neq\emptyset}}\omega^+(\hat{X}) - \sum_{\substack{\hat{X}\in\hat{\mathcal{X}}^+_{\Lambda}\\ \hat{X}\cap[\Gamma]_R\neq\emptyset}}\omega^+(\hat{X}).$$

Using symmetry,

$$|\log Q_c(0)| \leq 3\epsilon_c|[\Gamma]_R|. \tag{7.24}$$

For the derivative, a similar treatment gives

$$\left\| \frac{\mathrm{d}}{\mathrm{d}h} \log Q_c \right\|_{U_\Gamma} \leq 2\epsilon_c V(\Gamma) + 3\epsilon_c |[\Gamma]_R| \,. \tag{7.25}$$

Estimates (7.21) and (7.24) yield

$$2\beta V(\Gamma) |g_\Lambda^+(\Gamma)(0)| \leq 3(\epsilon_r + \epsilon_c) |[\Gamma]_R| \leq \delta_1(\beta)\beta\|\Gamma\| \tag{7.26}$$

where $\delta_1(\beta) := 3^{d+1}\beta^{-1}(\epsilon_r + \epsilon_e)\rho^{-1}$ (ρ is the Peierls constant). We get (7.13) by setting $\delta_2(\beta) := \beta^{-1}(\epsilon_r + \epsilon_e)$. □

We are now in position of computing derivatives of the functions $\varphi_\Lambda^+(\Gamma)$ and $u_\Lambda^+(\Gamma)$. The main ingredient is the following theorem, which appeared, in this form, in [Isakov2]. It is nothing but a stationary phase analysis applied to the Cauchy integral giving the k-th derivative at $z = 0$ of a function of the type $e^{-cz+bf(z)}$. The proof can be found in Appendix B.

Theorem 7.1. *Let $r > 0$, $F(z) = \exp(-cz + bf(z))$ where $1 \leq b \leq c$, and f is analytic in a disc $\{|z| < r\}$, taking real values on the real line, with a uniformly bounded derivative:*

$$\sup_{|z|<r} |f'(z)| \leq A < \frac{1}{25} \,. \tag{7.27}$$

There exists $k_0 = k_0(A)$ such that the following holds: define $k_+ = r(c - 2b\sqrt{A})$. For all integer $k \in [k_0, k_+]$ there exists $r_k \in (0, r)$ and $c_k > 0$ satisfying

$$\frac{k}{c + bA} \leq r_k \leq \frac{k}{c - bA} \,, \quad \frac{3}{10} \frac{1}{\sqrt{2\pi c r_k}} < \, c_k < \frac{1}{\sqrt{c r_k}} \,, \tag{7.28}$$

such that

$$F^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=r_k} \frac{F(z)}{z^{k+1}} \mathrm{d}z = k! \frac{c_k}{(-r_k)^k} F(-r_k) \,. \tag{7.29}$$

In particular, $(-1)^k F^{(k)}(0) > 0$. Moreover, if f satisfies the local condition

$$bf(0) \leq -\alpha r c \,, \tag{7.30}$$

with $\alpha \in (\log 2, 1)$, then for all $k \in [k_0, k_+]$ and A sufficiently small,

$$\big(\log(1 + F)\big)^{(k)}(0) = (1 + a \cdot e^{-\frac{1}{2}\zeta k}) F^{(k)}(0) \,, \tag{7.31}$$

where a is a bounded function of k, c, b and $\zeta = \zeta(\alpha) > 0$.

In Lemma 7.1, we have put $\varphi^+_\Lambda(\Gamma)$ in the form $e^{-cz+bf(z)}$. In order to satisfy (7.27), we must introduce a distinction among the contours. Consider the function $\delta_2(\beta)$ of (7.13).

Definition 7.1. *A contour $\Gamma \in \mathcal{C}^+(\Lambda)$ is* thin *if $|\Gamma| \le \frac{\delta_2(\beta)}{2} V(\Gamma)$, and* fat *if it is not thin.*

Now, any thin contour Γ satisfies, when β is large enough,

$$\Big\| \frac{\mathrm{d}}{\mathrm{d}h} g^+_\Lambda(\Gamma) \Big\|_{U_\Gamma} \le 2\delta_2(\beta) \equiv A(\beta) < \frac{1}{25}\,. \tag{7.32}$$

Lemma 7.2. *There exists k_0 such that when β is sufficiently large, the following holds. For all thin contour Γ, define*

$$k_+(\Gamma) := 2\beta V(\Gamma) R^*(V(\Gamma))(1 - 2\sqrt{A})\,. \tag{7.33}$$

Then for all integer $k \in [k_0, k_+(\Gamma)]$, we have

$$(-1)^k u^+_\Lambda(\Gamma)^{(k)}(0) \ge \frac{1}{10}\big(2\beta V(\Gamma) D_-\big)^k e^{-(1+\delta_1(\beta))\|\Gamma\|} \tag{7.34}$$

$$(-1)^k u^+_\Lambda(\Gamma)^{(k)}(0) \le 20\big(2\beta V(\Gamma) D_+\big)^k e^{-(1-\delta_1(\beta))\|\Gamma\|}\,, \tag{7.35}$$

where $\lim_{\beta\to\infty} D_\pm = 1$.

Proof. Let Γ be a thin contour. Consider $\varphi^+_\Lambda(\Gamma)$ in its exponentiated form (7.11). We apply Theorem 7.1 with $c = b = 2\beta V(\Gamma)$, $f = g^+_\Lambda(\Gamma) - \frac{1}{2}\frac{\|\Gamma\|}{V(\Gamma)}$, $r = R^*(V(\Gamma))$, and $A = A(\beta)$. (7.32) guarantees (7.27). There exists $r_k = r_k(\Gamma)$ and $c_k = c_k(\Gamma)$ such that

$$(-1)^k \varphi^+_\Lambda(\Gamma)^{(k)}(0) = k! \frac{c_k}{(r_k)^k} \varphi^+_\Lambda(\Gamma)(-r_k)\,. \tag{7.36}$$

Using the analyticity of $g^+_\Lambda(\Gamma)$ in U_Γ, we have with (7.28)

$$\begin{aligned}
\varphi^+_\Lambda(\Gamma)(-r_k) &= e^{-\beta\|\Gamma\|} e^{cr_k} e^{cg^+_\Lambda(\Gamma)(0)} e^{c(g^+_\Lambda(\Gamma)(-r_k) - g^+_\Lambda(\Gamma)(0))} \\
&\ge e^{-\beta\|\Gamma\|} e^{\frac{k}{1+A}} e^{-\delta_1\beta\|\Gamma\|} e^{-\frac{A}{1-A}k} \\
&= e^{-(1+\delta_1)\beta\|\Gamma\|} e^k e^{-\frac{2A}{1-A^2}k}\,.
\end{aligned}$$

Using Stirling's Formula and the estimates for r_k, c_k, we get

$$(-1)^k \varphi^+_\Lambda(\Gamma)^{(k)}(0) \ge \frac{1}{5}\big(2\beta V(\Gamma) D_-\big)^k e^{-(1+\delta_1)\beta\|\Gamma\|}\,, \tag{7.37}$$

where

$$D_-(\beta) = (1-A) e^{-\frac{2A}{1-A^2}}\,. \tag{7.38}$$

Using (7.12) we can satisfy (7.30):

$$
\begin{aligned}
bf(0) = 2\beta V(\Gamma) g^+_\Lambda(\Gamma)(0) - \beta\|\Gamma\| &\leq -(1-\delta_1)\beta\|\Gamma\| \\
&\leq -(1-\delta_1)2\beta V(\Gamma) R^*(V(\Gamma)) \qquad (7.39)\\
&= -(1-\delta_1) rc\,. \qquad (7.40)
\end{aligned}
$$

In (7.39) we used

$$
\|\Gamma\| \geq \frac{1}{K(V(\Gamma))} V(\Gamma)^{\frac{d}{d-1}} \geq 2V(\Gamma)\frac{\theta}{2K(V(\Gamma))V(\Gamma)^{\frac{1}{d}}} \geq 2V(\Gamma)R^*(V(\Gamma))\,.
$$

We can thus use (7.31) once β is large enough. This gives the lower bound (7.34). The upper bound is obtained similarly. □

7.2 Derivatives in a Finite Volume

In this section, we *fix* k large enough. When a thin contour satisfies $[k_0, k_+(\Gamma)] \ni k$ then $u^+_\Lambda(\Gamma)^{(k)}(0)$ can be estimated with Lemma 7.2. To characterise this class of contours, we introduce a k-dependent notion of size.

Definition 7.2. *Let $k \in \mathbb{N}$, $\epsilon' > 0$ small enough. A contour Γ is k-large if $V(\Gamma) \geq V_0(k)$ where*

$$
V_0(k) := \Big(\frac{K(\infty)(1+\epsilon')}{\theta\beta(1-2\sqrt{A})}k\Big)^{\frac{d}{d-1}}, \qquad (7.41)
$$

where $K(\infty)$ was defined in Lemma 4.10. Γ is k-small if $V(\Gamma) < V_0(k)$.

Let $N_0(\epsilon')$ be such that for all $N \geq N_0(\epsilon')$ (see Lemma 6.2),

$$
\frac{1}{(1+\epsilon')}\frac{\theta}{2K(\infty)N^{\frac{1}{d}}} \leq R^*(N) \leq \frac{\theta}{2K(\infty)N^{\frac{1}{d}}}\,. \qquad (7.42)
$$

Let $k_- = k_-(\epsilon', \gamma)$ be such that when $k \geq k_-$ then $V_0(k) \geq N_0(\epsilon')$. This definition implies that when $k \geq k_-$, we have for all k-large contour Γ

$$
k_+(\Gamma) = 2\beta V(\Gamma)(1-2\sqrt{A})R^*(V(\Gamma)) \geq \frac{\theta\beta(1-2\sqrt{A})}{K(\infty)(1+\epsilon')}V(\Gamma)^{\frac{d-1}{d}} \geq k\,. \qquad (7.43)
$$

That is, the k-th derivative of a k-large thin contour can be studied with Lemma 7.2. The dependence of k_- on γ comes from the bound $K(\infty) \geq c_-\gamma$. We therefore have $\lim_{\gamma\searrow 0} k_- = +\infty$.

Proposition 7.1. *Let θ be close to 1, β large enough. There exists a constant $C_1 > 0$ and an unbounded increasing sequence of integers $k_1, k_2, \dots$ such that for large N, we have whenever Λ is sufficiently large,*

$$\frac{(-1)^{k_N}}{|\Lambda|} \frac{\mathrm{d}^{k_N}}{\mathrm{d}h^{k_N}} \sum_{\Gamma \in \mathcal{C}^+(\Lambda)} u_\Lambda^+(\Gamma)\Big|_{h=0} \geq \big(C_1 K(\infty)^{\frac{d}{d-1}} \beta^{-\frac{1}{d-1}}\big)^{k_N} k_N!^{\frac{d}{d-1}} . \tag{7.44}$$

Proof. Fix $\epsilon > 0$ small and consider the sequence $(\Gamma_N)_{N\geq 1}$ of Lemma 4.10. We have $\lim_{N\to\infty} V(\Gamma_N) = +\infty$ and when N is large enough,

$$(1-\epsilon)K(\infty) \leq \frac{V(\Gamma_N)^{\frac{d-1}{d}}}{\|\Gamma_N\|} \leq (1+\epsilon)K(\infty) . \tag{7.45}$$

The sequence $(k_N)_{N\geq 1}$ is defined such that the contribution from the contour Γ_N to $\mathfrak{p}_{\gamma,\Lambda}^{+(k_N)}(0)$ is close to maximal. Let

$$k_N := \Big\lfloor \frac{d-1}{d} \beta \|\Gamma_N\| \Big\rfloor . \tag{7.46}$$

Since $\lim_{N\to\infty} V(\Gamma_N) = +\infty$, we have $\lim_{N\to\infty} k_N = +\infty$. From now on we consider N large enough so that (7.45) and (7.48) hold and $k_N \geq \max\{k_0, k_-\}$. When considering the k_N-th derivative, we use the following decomposition:

$$\sum_{\Gamma \in \mathcal{C}^+(\Lambda)} = \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ k_N-\text{large, thin}}} + \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ k_N-\text{small, thin}}} + \sum_{\substack{\Gamma \in \mathcal{C}^+(\Lambda) \\ \text{fat}}} \tag{7.47}$$

We show that the dominant term comes from Γ_N, which belongs to the first sum, and that the two other sums are negligible. To see that Γ_N appears in the first sum, we first show that Γ_N is k_N-large. Indeed, if θ is close to 1 and $\epsilon, \epsilon', A(\beta)$ are small,

$$\begin{aligned} V_0(k_N) &\leq \Big(\frac{K(\infty)(1+\epsilon')}{\theta(1-2\sqrt{A})} \frac{d-1}{d} \|\Gamma_N\|\Big)^{\frac{d}{d-1}} \\ &\leq \Big(\frac{1}{\theta(1-2\sqrt{A})} \frac{1+\epsilon'}{1-\epsilon} \frac{d-1}{d}\Big)^{\frac{d}{d-1}} V(\Gamma_N) \leq V(\Gamma_N) . \end{aligned}$$

Then we show that Γ_N is thin:

$$\frac{|\Gamma_N|}{V(\Gamma_N)} \leq \frac{1}{\rho} \frac{\|\Gamma_N\|}{V(\Gamma_N)} \leq \frac{1}{\rho K(\infty)(1-\epsilon)} \frac{1}{V_0(k_N)^{\frac{1}{d}}} \leq \frac{1}{2} \delta_2(\beta) . \tag{7.48}$$

Finally, we assume Λ is large enough in order to contain at least $a|\Lambda|$ translates of Γ_N, $a > 0$. Then we apply Lemma 7.2 to $u_\Lambda^+(\Gamma_N)$. Using (7.45),

$$\begin{aligned} V(\Gamma_N)^{k_N} e^{-(1+\delta_1)\beta\|\Gamma_N\|} &\geq \big((1-\epsilon)K(\infty)\|\Gamma_N\|\big)^{\frac{d}{d-1}k_N} e^{-(1+\delta_1)\beta\|\Gamma_N\|} \\ &\geq \Big((1-\epsilon)K(\infty)\frac{d}{d-1}\frac{1}{\beta}k_N\Big)^{\frac{d}{d-1}k_N} e^{-(1+\delta_1)\frac{d}{d-1}(k_N+1)} \\ &\geq c(k_N)K(\infty)^{\frac{d}{d-1}k_N}\beta^{-\frac{d}{d-1}k_N}\Big[\frac{d}{d-1}(1-\epsilon)e^{-\delta_1}\Big]^{\frac{d}{d-1}k_N} k_N!^{\frac{d}{d-1}}\,, \end{aligned} \tag{7.49}$$

where $c(k_N) \geq C_3 k_N^{-\frac{1}{2}}$ and we used Stirling's Formula. Since

$$(-1)^{k_N} u_\Lambda^+(\Gamma)^{(k_N)}(0) \geq 0 \tag{7.50}$$

for all k_N-large thin contour, we can obtain a lower bound on the first sum of (7.47) by using only the contributions coming from the translates of Γ_N. We get

$$\frac{(-1)^{k_N}}{|\Lambda|}\frac{\mathrm{d}^{k_N}}{\mathrm{d}h^{k_N}} \sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ k_N-\text{large, thin}}} u_\Lambda^+(\Gamma)\Big|_{h=0} \geq$$

$$\frac{ac(k_N)}{20}2^{k_N}K(\infty)^{\frac{d}{d-1}k_N}\beta^{-\frac{1}{d-1}k_N}\Big[\frac{d}{d-1}(1-\epsilon)e^{-\delta_1}D_-\Big]^{\frac{d}{d-1}k_N} k_N!^{\frac{d}{d-1}}\,. \tag{7.51}$$

Consider now a k_N-small thin contour, i.e. $R^*(V(\Gamma)) \geq R^*(V_0(k_N))$. Using Cauchy's Formula with a disc of radius $R^*(V_0(k_N))$ centered at $h = 0$,

$$|u_\Lambda^+(\Gamma)^{(k_N)}(0)| \leq k_N!\Big(\frac{1}{R^*(V_0(k_N))}\Big)^{k_N}\|u_\Lambda^+(\Gamma)\|_{U_\Gamma}\,. \tag{7.52}$$

Lemma 7.3. *Setting* $\alpha_1 = \alpha_1(\theta,\beta) := \rho^{-1}(1-\theta(1+A(\beta))-\delta_1(\beta))$. *If* β *is large enough, we have* $\alpha_1 > 0$ *and the bound*

$$\|u_\Lambda^+(\Gamma)\|_{U_\Gamma} \leq \frac{e^{-\beta\alpha_1|\Gamma|}}{1-e^{-\beta\alpha_1|\Gamma|}}\,. \tag{7.53}$$

Proof. Using (7.11), (7.12) and (7.32),

$$\|\varphi_\Lambda^+(\Gamma)\|_{U_\Gamma} \leq \sup_{h\in U_\Gamma} e^{-\beta(1-\delta_1)\|\Gamma\|}e^{2\beta(1+A)|\mathrm{Re}\,h|V(\Gamma)} \leq e^{-\alpha_1\beta|\Gamma|} < 1\,, \tag{7.54}$$

where we used the definition of the radius of analyticity:

$$\sup_{h\in U_\Gamma}|h|V(\Gamma) \leq R^*(V(\Gamma))V(\Gamma) \leq R(V(\Gamma))V(\Gamma) \leq \frac{\theta}{2}\|\Gamma\|\,. \tag{7.55}$$

The proof finishes by using the Taylor expansion of $\log(1+x)$. □

A standard Peierls estimate implies, when β is large, the existence of a number C_4 such that

$$\sum_{\Gamma\in\mathcal{C}^+(\Lambda)} e^{-\beta\alpha_1|\Gamma|} \leq C_4|\Lambda|\,. \tag{7.56}$$

Using Stirling's Formula, it easy to see that $k_N!k_N^{\frac{1}{d-1}k_N} \leq k_N!^{\frac{d}{d-1}}e^{\frac{1}{d-1}k_N}$. The contribution from the k_N-small contours is then bounded by

$$\frac{1}{|\Lambda|}\Big|\frac{\mathrm{d}^{k_N}}{\mathrm{d}h^{k_N}}\sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ k_N-\text{small, thin}}} u_\Lambda^+(\Gamma)\Big|_{h=0} \leq$$
$$C_5 2^{k_N}K(\infty)^{\frac{d}{d-1}k_N}\beta^{-\frac{1}{d-1}k_N}\Big[e^{\frac{1}{d-1}}\Big(\frac{1+\epsilon'}{\theta}\Big)^{\frac{d}{d-1}}\Big(\frac{1}{1-2\sqrt{A}}\Big)^{\frac{1}{d-1}}\Big]^{k_N}k_N!^{\frac{d}{d-1}} \tag{7.57}$$

Now, compare (7.57) with (7.51). Comparing the square brackets in each of these equations shows that the lower bound obtained with the k_N-large contours is dominant once θ is close to 1, ϵ,ϵ' are small, and if β is large enough. This follows from the inequality

$$\frac{d}{d-1} > e^{\frac{1}{d}}\,,\quad \forall\, d\geq 2\,, \tag{7.58}$$

which follows from the following computation:

$$\begin{aligned} d\big(\mathrm{e}^{\frac{1}{d}}-1\big) &= d\big(\mathrm{e}^{\frac{1}{d}}-1-\frac{1}{d}+\frac{1}{d}\big) = \sum_{n\geq 2}\frac{1}{n!}\Big(\frac{1}{d}\Big)^{n-1}+1\\ &= 1+\sum_{n\geq 1}\frac{1}{(n+1)!}\Big(\frac{1}{d}\Big)^{n}\\ &< 1-\frac{1}{2d}+\sum_{n\geq 1}\frac{1}{n!}\Big(\frac{1}{d}\Big)^{n} = \mathrm{e}^{\frac{1}{d}}-\frac{1}{2d}\,. \end{aligned}$$

We are then left with the contribution of the fat contours. We can use a Cauchy bound

$$\begin{aligned} \Big|\frac{\mathrm{d}^k}{\mathrm{d}h^k}u_\Lambda^+(\Gamma)\Big|_{h=0} &\leq k!\Big(\frac{1}{R^*(V(\Gamma))}\Big)^k\|u_\Lambda^+(\Gamma)\|_{U_\Gamma}\\ &\leq k!\Big(\frac{2K(1)}{\theta}\Big)^k V(\Gamma)^{\frac{k}{d}}\frac{e^{-\beta\alpha_1|\Gamma|}}{1-e^{-\beta\alpha_1|\Gamma|}}\\ &\leq k!\Big(\frac{2K(1)}{\theta}\Big(\frac{2}{\delta_2}\Big)^{\frac{1}{d}}\Big)^k|\Gamma|^{\frac{k}{d}}\frac{e^{-\beta\alpha_1|\Gamma|}}{1-e^{-\beta\alpha_1|\Gamma|}}\,. \end{aligned}$$

Then a Peierls estimate leads to

$$\sum_{\Gamma\in\mathcal{C}^+(\Lambda)} |\Gamma|^{\frac{k}{d}} e^{-\alpha_1\beta|\Gamma|} \leq |\Lambda| \sum_{L\geq 1} L^{\frac{k}{d}} e^{-\alpha_1'\beta L} \leq |\Lambda|(\alpha_1'\beta)^{-\frac{k}{d}}\Gamma\big(\frac{k}{d}+1\big)\,, \tag{7.59}$$

where $\Gamma(x)$ is the Gamma-function. Using Stirling's Formula, it is then easy to show that the contribution from the fat contours is bounded by

$$\frac{1}{|\Lambda|}\Big|\frac{\mathrm{d}^k}{\mathrm{d}h^k}\sum_{\substack{\Gamma\in\mathcal{C}^+(\Lambda)\\ \text{fat}}} u^+_\Lambda(\Gamma)\Big|_{h=0} \leq \big(K(1)\beta^{-\frac{1}{d}}D(k)\big)^k k!^{\frac{d}{d-1}}\,, \tag{7.60}$$

where $\lim_{k\to\infty} D(k) = 0$. The fat contours can thus always be ignored. This finishes the proof of Proposition 7.1 □

With (5.58), we get the lower bound, for a large enough box Λ,

$$|\mathfrak{p}^{+(k_N)}_{\gamma,\Lambda}(0)| \geq \big(C_1 K(\infty)^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\big)^{k_N} k_N!^{\frac{d}{d-1}} - C_r^{k_N} k_N! \tag{7.61}$$

$$\geq \big(C_-\gamma^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\big)^{k_N} k_N!^{\frac{d}{d-1}} - C_r^{k_N} k_N!\,. \tag{7.62}$$

We used the lower bound $K(\infty) \geq c_-\gamma$ from Lemma 4.10. Notice that we could extract the contribution of the translates of Γ_N to $\mathfrak{p}^{+(k_N)}_{\gamma,\Lambda}(0)$ *without* knowing its explicit shape. This is where our formulation of the isoperimetric problems differs from the one of Isakov. Notice also that the lower bound (7.62) shows how non-analyticity is detected in *finite* volumes.

7.3 Proof of Theorem 3.4

To extend the bounds we have on $\mathfrak{p}^{+(k_N)}_{\gamma,\Lambda}(0)$ to the infinite volume limit, we first show that in the strip U_+ the derivatives of the pressure are uniformly bounded.

Lemma 7.4. *Let β be large enough. There exists $C_+ > 0$ such that for all $k \geq 2$,*

$$\sup_\Lambda \|\mathfrak{p}^{+(k)}_{\gamma,\Lambda}\|_{U_+} \leq \big(C_+\gamma^{\frac{d}{d-1}}\beta^{-\frac{1}{d-1}}\big)^k k!^{\frac{d}{d-1}} + C_r^k k!\,. \tag{7.63}$$

Proof. Like in Section 6.4, we can fix $\theta := \frac{1}{2}$. The term $C_r^k k!$ comes from (5.58). Consider $u^+_\Lambda(\Gamma)$ and the representation (7.14) of $\varphi^+_\Lambda(\Gamma)$. From Lemma 7.1, $\varphi^+_\Lambda(\Gamma)$ is analytic in U_Γ. From Proposition 6.2 and Lemma 6.4, it is also analytic in U_+, i.e. in $U_+\cup U_\Gamma$. Proceeding like in the proof of Lemma 7.1, we get

$$\Big\|\frac{\Theta^+_{i(\Gamma)}(\mathrm{ext}_\Lambda\Gamma;\sigma_\Gamma)\Theta^+(\mathrm{int}\Gamma;-\sigma_\Gamma)}{\Theta^+_{i(\Gamma)}(\Lambda)}\Big\|_{U_+} \leq \sup_{h\in U_+} e^{-\beta\mathrm{Re}\,h|\Gamma|}e^{3(\epsilon_r+\epsilon_c)|[\Gamma]_R|}$$

$$= e^{3(\epsilon_r+\epsilon_c)|[\Gamma]_R|}\,.$$

Assume $3^{d+1}(\epsilon_r + \epsilon_c) \leq \frac{1}{3}$. Using (6.59),

$$\|\varphi_\Lambda^+(\Gamma)\|_{U_+} \leq e^{-\beta\|\Gamma\|} e^{\beta h_+ |\Gamma|} e^{|\Gamma|} \leq e^{-\alpha_2 \beta |\Gamma|} < 1\,. \tag{7.64}$$

Notice that unlike in (7.54), α_2 in independent of θ. This implies that $u_\Lambda^+(\Gamma)$ is also analytic in $U_+ \cup U_\Gamma$. Set $\alpha_3 = \min\{\alpha_1, \alpha_2\}$. Using a disc of radius $R^*(V(\Gamma))$ around each $h \in U_+$, we have

$$\begin{aligned}
\|u_\Lambda^+(\Gamma)^{(k)}\|_{U_+} &\leq k!\Big(\frac{1}{R^*(V(\Gamma))}\Big)^k \|u_\Lambda^+(\Gamma)\|_{U_+ \cup U_\Gamma} \\
&\leq k!\Big(\frac{2K(1)}{\theta}\Big)^k V(\Gamma)^{\frac{k}{d}} \frac{e^{-\beta\alpha_3|\Gamma|}}{1 - e^{-\beta\alpha_3|\Gamma|}} \\
&\leq k!\Big(\frac{2K(1)}{\theta l^{\frac{1}{d-1}}}\Big)^k |\Gamma|^{\frac{k}{d-1}} \frac{e^{-\beta\alpha_3|\Gamma|}}{1 - e^{-\beta\alpha_3|\Gamma|}}\,.
\end{aligned}$$

We used the isoperimetric inequality of Lemma 4.9. Remember that $K(1) \leq c_+\gamma$ (Lemma 4.10), and that $l = \nu\gamma^{-1}$. The proof finishes like for the upper bound on fat contours:

$$\sum_{\Gamma \in \mathcal{C}^+(\Lambda)} |\Gamma|^{\frac{k}{d-1}} e^{-\beta\alpha_3|\Gamma|} \leq |\Lambda| \sum_{L \geq 1} L^{\frac{k}{d-1}} e^{-\beta\alpha_3' L}\,. \tag{7.65}$$

□

Let $\mathfrak{p}_\gamma^{(k),\leftarrow}(h)$ denote the k-th right directional derivative of $\mathfrak{p}_\gamma$ at h.

Corollary 7.1. *For all $h' \in U_+ \cup \{\operatorname{Re} h = 0\}$ and for all $k \in \mathbb{N}$,*

$$\mathfrak{p}_\gamma^{(k),\leftarrow}(h') = \lim_{\Lambda \nearrow \mathbf{Z}^d} \mathfrak{p}_{\gamma,\Lambda}^{+(k)}(h') = \lim_{h \searrow h'} \mathfrak{p}_\gamma^{(k)}(h)\,. \tag{7.66}$$

Proof. We show (7.66) for $k = 1$. By definition,

$$\begin{aligned}
\mathfrak{p}_\gamma^{(1),\leftarrow}(h') &= \lim_{\delta \searrow 0} \frac{\mathfrak{p}_\gamma(h' + \delta) - \mathfrak{p}_\gamma(h')}{\delta} \\
&= \lim_{\delta \searrow 0} \lim_{\Lambda \nearrow \mathbf{Z}^d} \frac{\mathfrak{p}_{\gamma,\Lambda}^+(h' + \delta) - \mathfrak{p}_{\gamma,\Lambda}^+(h')}{\delta} \\
&= \lim_{\delta \searrow 0} \lim_{\Lambda \nearrow \mathbf{Z}^d} \big(\mathfrak{p}_{\gamma,\Lambda}^{+(1)}(h') + \frac{1}{2!}\mathfrak{p}_{\gamma,\Lambda}^{+(2)}(h(\delta))\delta\big)\,,
\end{aligned}$$

where $\lim_{\delta \searrow 0} h(\delta) = h'$. The following lemma will allow to permute the limits $\lim_{\delta \searrow 0}$ and $\lim_{\Lambda \nearrow \mathbf{Z}^d}$.

Lemma 7.5. *Let, for all $N \in \mathbb{N}$, $\delta > 0$, $b_N(\delta) = a_N + c_N(\delta)$, such that $|c_N(\delta)| \leq D\delta$ uniformly in N, and $\lim_{N\to\infty} b_N(\delta) = b(\delta)$ exists. Then $\lim_{N\to\infty} a_N$ and $\lim_{\delta \searrow 0} b(\delta)$ exist and are equal.*

Proof. We first show that $\lim_{\delta\searrow 0} b(\delta)$ exists. Let (δ_k) be any sequence $\delta_k > 0$ such that $\lim_{k\to\infty}\delta_k = 0$. Then we have

$$|b(\delta_k) - b(\delta_{k'})| = |\lim_{N\to\infty}(c_N(\delta_k) - c_N(\delta_{k'}))| \leq D(\delta_k + \delta_{k'})\,, \tag{7.67}$$

and so $\lim_{k\to\infty} b(\delta_k)$ exists. Fix $\epsilon > 0$. There exists $N_{\epsilon,\delta}$ such that if $N \geq N_{\epsilon,\delta}$ then $|b_N(\delta) - b(\delta)| \leq \epsilon$. We then have

$$b(\delta) - \epsilon - D\delta \leq \liminf_{N\to\infty} a_N \leq \limsup_{N\to\infty} a_N \leq b(\delta) + \epsilon + D\delta\,, \tag{7.68}$$

which finishes the proof, once we take $\epsilon \to 0$, $\delta \to 0$. □

Using the fact that the second derivative is uniformly bounded on U_+ (Lemma 7.4) shows the first equality in (7.66). For the second, we only need to consider the case where $h' = 0$.

$$\begin{aligned}
\mathfrak{p}_\gamma^{(1),\leftarrow}(0) &= \lim_{\delta\searrow 0}\frac{\mathfrak{p}_\gamma(\delta) - \mathfrak{p}_\gamma(0)}{\delta} \\
&= \lim_{\delta\searrow 0}\Big[\frac{\mathfrak{p}_\gamma(\delta) - \mathfrak{p}_\gamma(\frac{\delta}{2})}{\delta} + \frac{\mathfrak{p}_\gamma(\frac{\delta}{2}) - \mathfrak{p}_\gamma(0)}{\delta}\Big] \\
&= \big(\lim_{\delta\searrow 0}\frac{1}{2}\mathfrak{p}_\gamma^{(1)}(h(\delta))\big) + \frac{1}{2}\mathfrak{p}_\gamma^{(1),\leftarrow}(0)\,,
\end{aligned}$$

where $h(\delta) \in [\frac{\delta}{2}, \delta]$ and $\lim_{\delta\searrow 0} h(\delta) = 0$. This shows

$$\mathfrak{p}_\gamma^{(1),\leftarrow}(0) = \lim_{\delta\searrow 0}\mathfrak{p}_\gamma^{(1)}(h(\delta))\,, \tag{7.69}$$

which extends easily to any sequence $h \searrow 0$, since derivatives of any order are uniformly bounded on U_+. □

We can then complete the proof of Theorem 3.4:

Proof of Theorem 3.4: The bounds on $\mathfrak{p}_{\gamma,\Lambda}^{(k)}(0)$ of (7.62) and Lemma 7.4 extend to the thermodynamic limit using Corollary 7.1. □

7.4 Conclusion

Our analysis has lead to the following representation of the pressure for $h \geq 0$:

$$\mathfrak{p}_\gamma(h) = \mathfrak{p}_{r,\gamma}^{+}(h) + \mathfrak{s}_\gamma^{+}(h)\,. \tag{7.70}$$

As we have seen in Chapter 5, the restricted pressure $\mathfrak{p}_{r,\gamma}^{+}$, which describes a homogeneous phase with positive magnetisation, behaves analytically at $h = 0$. On the other hand, $\mathfrak{s}_\gamma^{+}$ is the singular contribution to the pressure: it contains the contributions from contours (droplets) of any possible sizes, and is responsible for the non-analytic behaviour of the pressure at $h = 0$. Nevertheless, the contribution of $\mathfrak{s}_\gamma^+$ to the pressure is essentially zero when γ is small. Indeed, $\mathfrak{s}_\gamma^+$ can be expressed as a sum over clusters of chains, and each chain contains at least one contour. Since the length of a contour is bounded below by the size of a cube $C^{(l)}$, we have

$$\|\mathfrak{s}_\gamma^+\|_{U_+} \leq a e^{-b\beta\gamma^{-d}}, \tag{7.71}$$

where $a, b > 0$ are constants. Combined with the Lebowitz-Penrose Theorem (see Chapter 3), the bound (7.71) implies, for $h \geq 0$,

$$\mathfrak{p}_0(h) = \lim_{\gamma \searrow 0} \mathfrak{p}_{r,\gamma}^+(h) = \sup_{m \geq 0} \big(hm - \mathfrak{f}_{MF}(m) \big). \tag{7.72}$$

From this last expression, the analytic continuation of the pressure, in the van der Waals Limit, can be understood easily: for $h > 0$, $hm - \mathfrak{f}_{MF}(m)$ has a unique global maxima at $m^*(h, \beta) > 0$. When $h < 0$ this maxima is only local, but provides the analytic continuation of $\mathfrak{p}_0$ at $h = 0$. The identity (7.72) shows that the constraint on the local magnetisation, in $\mathfrak{p}_{r,\gamma}^+$, has the effect of always selecting the maxima $m^*(h, \beta)$, which is global when $h > 0$ and local when $h < 0$.

When $\gamma > 0$, this scenario breaks down: $\mathfrak{s}_\gamma^+ \neq 0$ and large droplets of the $-$ phase are stable at $h = 0$. This yields a contribution $k!^{\frac{d}{d-1}}$ to the k-th derivative of the pressure. This shows that the analytic continuation, as obtained in van der Waals and mean field theories, is possible only when the system is assumed to be homogeneous in space. On the other hand, when the system is allowed to condense, there is creation of a non-analytic singularity at the condensation point.

Chapter 8

Discussion and Open Problems

This chapter contains concluding remarks and an account of some open problems that deserve further attention in the future. In Section 8.1, we also mention a few other results, stated without proof, obtained during our work.

8.1 Overview

The results we have obtained in this thesis can be summarized as follows:

We have successfully extended the technique of [Isakov1] to the class of general two phase models of Pirogov-Sinai Theory (Theorem 1.2), and made a precise link with the mean field behaviour, using Kac potentials and the van der Waals Limit (Theorem 1.3). Our analysis shows that the finiteness of the range of interaction is a sufficient condition for creating a singularity at which no analytic continuation is possible. This confirms the droplet mechanism proposed by Andreev, Fisher and Langer, which we presented in Section 1.2.2. Configurations of finite range models - as opposed to the mean field approximation in which the geometry of the system plays no role - can always be described with contours. At the transition point, contours of all sizes are stable, yielding the behaviour $\sim k!^{\frac{d}{d-1}}$ for the large derivatives of the pressure.

8.1.1 Generic Features of First Order Phase Transitions

The analysis of large order derivatives of the pressure reveals other features of first order phase transitions, which we briefly describe.

A Signature of Non-Analyticity in Finite Volumes. It is known that phase transitions occur only in the thermodynamic limit. The meaning of this statement can be understood via the study of the derivatives of the pressure. Let $\mathfrak{p}_\Lambda$ denote the pressure of a lattice system with finite range interactions, in a finite box Λ. Consider the Ising Model for simplicity. Since this system is finite, $\mathfrak{p}_\Lambda$ is analytic at $h = 0$. On the other hand, the analysis shows that for a fixed integer k, the box Λ can be taken sufficiently large so that

$$|\mathfrak{p}_\Lambda^{(k)}(0)| \geq C^k k!^{\frac{d}{d-1}}, \tag{8.1}$$

where the constant C is uniform in the volume. We have then seen that (8.1) implies the same bound in the thermodynamic limit:

$$|\mathfrak{p}^{(k)}(0^+)| \geq C^k k!^{\frac{d}{d-1}}. \tag{8.2}$$

Clearly, the thermodynamic limit is essential to create a *real* singularity, i.e. to obtain (8.2) for an unbounded increasing sequence $k_1, k_2, \dots$, yielding a Taylor expansion of the pressure with radius of convergence equal to zero. Nevertheless, the behaviour $\sim k!^{\frac{d}{d-1}}$ can already be detected in *finite* systems, as (8.1) shows. In this sense, the study of large finite systems suffices to predict the singularity which occurs in the thermodynamic limit.

A Signature of Non-Analyticity Outside the Transition Point. In the same spirit, we wish to emphasize another point. Namely, the lower bound (8.2) holds for an infinite number of integers k only if the derivative is evaluated *at* the point $h = 0$; we want to show that this behaviour can actually be detected in a *neighbourhood* of $h = 0$.

We have seen that the pressure is analytic outside the transition point. For the Ising Model, this is a consequence of Lee and Yang's theorem. Application of the Cauchy Formula then yields, for all $h \neq 0$,

$$|\mathfrak{p}^{(k)}(h)| \leq {B_0}^k k!, \quad B_0 = B_0(h) > 0. \tag{8.3}$$

An interesting question is to know if, for a fixed integer k, a crossover between the two behaviours - (8.3) at $h \neq 0$ and (8.2) at $h = 0$ - can be explicited. We state a theorem which answers partially the question.

Theorem 8.1. *Consider the pressure of the Ising Model, $\mathfrak{p} = \mathfrak{p}(h)$. Let β be large enough. Then, for all large enough integer k, there exists $h(k) > 0$ such that for all $h \in [0, h(k)]$, ($h(k) \searrow 0$ when $k \nearrow +\infty$)*

$$|\mathfrak{p}^{(k),\leftarrow}(h)| \geq C^k k!^{\frac{d}{d-1}}. \tag{8.4}$$

The bounds (8.3) and (8.4) thus allow, to a certain extent, "measuring" the distance from a point $h \neq 0$ to the transition point $h = 0$. The proof of Theorem 8.1 follows the same lines as the proofs given in the rest of the thesis. In particular, one must apply Corollary B.1 with $t \neq 0$. Combined with the remarks of the previous paragraph, we conclude that the phase transition can, in principle, be detected *for finite volumes, outside the transition point.*

Borel-Summability at the Transition Point. We have obtained bounds on the derivatives of the pressure that are uniform in the strip $0 < \mathrm{Re}\, h < \epsilon$ (these hold for the Ising model as well as for the Kac model):

$$\sup_{0<\mathrm{Re}\, h<\epsilon} |\mathfrak{p}^{(k)}(h)| \leq C^k k!^{\frac{d}{d-1}}, \quad \forall k \in \mathbb{N}. \tag{8.5}$$

We will now see that this uniformity guarantees that the coefficients $\frac{1}{k!}\mathfrak{p}^{(k)}(0^+)$ provide an asymptotic expansion of the pressure at $h = 0$, and, moreover, that this asymptotic expansion allows to reconstruct the function via Borel summation.

By the Theorem of Yang-Lee, the pressure is analytic in $\mathrm{Re}\, h > 0$ [1]. Consider any real $h_0 > 0$, and express the pressure as a finite Taylor expansion around h_0: for $h \in \mathbb{C}$ with $\mathrm{Re}\, h > 0$,

$$\mathfrak{p}(h) = \sum_{k=0}^{N-1} \frac{1}{k!}\mathfrak{p}^{(k)}(h_0)(h-h_0)^k + R_N(h;h_0), \tag{8.6}$$

where the rest $R_N(h;h_0)$ is given by

$$R_N(h;h_0) = \frac{1}{N!}\mathfrak{p}^{(N)}(\tilde{h})(h-h_0)^N, \quad \tilde{h} = \tilde{h}(h,h_0) \in [h_0, h]. \tag{8.7}$$

Using the uniform bound (8.5), we have

$$\left|\mathfrak{p}(h) - \sum_{k=0}^{N-1} \frac{1}{k!}\mathfrak{p}^{(k)}(h_0)(h-h_0)^k\right| = |R_N(h;h_0)| \leq C^N N!^{\frac{1}{d-1}} |h-h_0|^N \tag{8.8}$$

We can then take the limit $h_0 \searrow 0$ in this last inequality, and get

$$\left|\mathfrak{p}(h) - \sum_{k=0}^{N-1} \frac{1}{k!}\mathfrak{p}^{(k)}(0^+)h^k\right| \leq C^N N!^{\frac{1}{d-1}} |h|^N. \tag{8.9}$$

This shows that $\sum_k \frac{1}{k!}\mathfrak{p}^{(k)}(0^+)h^k$ provides an asymptotic expansion of the pressure (see [Rem1], p.296). Moreover, the hypotheses of the Watson-Sokal Lemma [So] are satisfied, and the pressure $\mathfrak{p}$ can be reconstructed with the coefficients $\mathfrak{p}^{(k)}(0^+)$. Namely, the Borel sum

$$B(t) := \sum_{k\geq 0} \frac{1}{k!^2}\mathfrak{p}^{(k)}(0^+)t^k \tag{8.10}$$

converges in the disc $|t| < C^{-1}$, has analytic continuation to the strip-like region $\{t \in \mathbb{C} : d(t, \mathbb{R}^+) \leq C^{-1}\}$, and $\mathfrak{p}$ can be represented by the absolutely convergent integral

$$\mathfrak{p}(h) = \frac{1}{h}\int_0^\infty e^{-\frac{t}{h}} B(t)\mathrm{d}t, \quad \mathrm{Re}\, h > 0. \tag{8.11}$$

[1] In the general case of two phase models, analyticity in a half space can be obtained via the methods presented in Appendix A.

Temperature Driven Transition in the Potts Model. The proofs of our results have always used a contour representation of the underlying spin system. In fact, any two phase model that can be represented with contours is expected to present the same non-analytic behaviour. A typical example is the temperature driven transition of the Potts Model at large q. This model is defined as follows: for each $i \in \mathbf{Z}^d$, let σ_i be a spin taking values in the set $\{1, 2, \ldots, q\}$. Define the (formal) hamiltonian

$$H = -\sum_{\langle i,j\rangle} \delta_{\sigma_i,\sigma_j}\,, \tag{8.12}$$

where $\delta_{x,y} = 1$ if $x = y$ and otherwise equals zero (the sum is over nearest neighbours). Consider the pressure $\mathfrak{p} = \mathfrak{p}(\beta)$ of this model, seen as a function of the inverse temperature β.

Often, the Potts Model is studied via its Fortuin-Kasteleyn representation, in which critical properties can be studied with percolation arguments [FK]. Using the contour representation of the Fortuin-Kasteleyn representation [LMMRS], the Pirogov-Sinai Theory allows, when q is sufficiently large, to study the system in a neighbourhood of its transition temperature $\beta_c(q)$. Our technique also applies to this case, and following the same proof as for Theorem 1.2, we can show:

Theorem 8.2. *For sufficiently large q, the free energy $\mathfrak{f}(\beta)$ of the Potts model has a first order phase transition at $\beta_c(q)$; it is analytic in β on $\{\beta < \beta_c(q)\}$ and $\{\beta > \beta_c(q)\}$, but has no analytic continuation from $\{\beta < \beta_c(q)\}$ to $\{\beta > \beta_c(q)\}$ accross $\beta_c(q)$ (or vice-versa).*

More than Two Phases. An interesting situation is when more than two phases are present. The Blume-Capel Model is the prototype of a three phase model, in which each spin can take three values, $s_i \in \{-1, 0, +1\}$. The hamiltonian has the form

$$H = \sum_{\langle i,j\rangle}(s_i - s_j)^2 - \mu_1 \sum_i s_i^2 - \mu_2 \sum_i s_i\,. \tag{8.13}$$

We have depicted the phase diagram of this model on Figure 8.1. The Pirogov-Sinai Theory allows to obtain analyticity of the pressure $\mathfrak{p} = \mathfrak{p}(\underline{\mu})$, $\underline{\mu} = (\mu_1, \mu_2)$, with respect to either of the fields μ_1, μ_2, in the pure phase regions U_+, U_0, U_-. Far from the triple point, the system effectively behaves as a two phase model when moving along a path that crosses a coexistence line. Theorem 1.2 thus shows that the pressure has no analytic continuation along such paths (see Figure 8.1). A weakness of this result is that the range of temperature for which the result holds decreases when the path approaches the triple point.

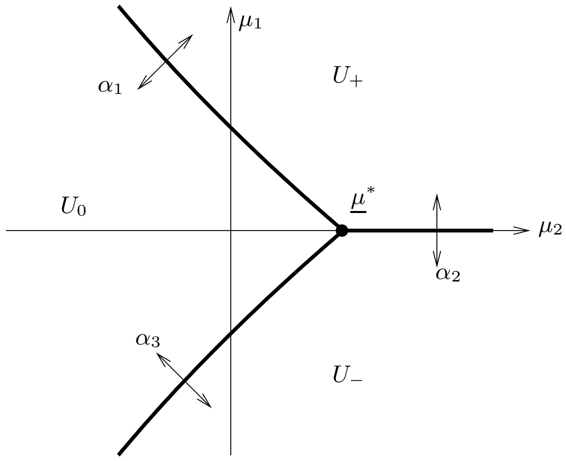

Figure 8.1: The low temperature phase diagram of the Blume-Capel Model. The triple point $\underline{\mu}^* = (\mu_2^*, 0)$ is the point of maximal coexistence. The regions U_+, U_0, U_- are regions of pure phases, and the lines $U_+ \cap U_0$, $U_0 \cap U_-$, $U_+ \cap U_-$ are the lines of coexistence. The paths $\alpha_1, \alpha_2, \alpha_3$ are the paths along which Theorem 1.2 applies.

For paths that go *through* the triple point, a different analysis must be done, treating the three phases equivalently. Our result for a particular class of paths is the following [2]:

Theorem 8.3. *Let β be sufficiently large. Then the pressure $\mathfrak{p} = \mathfrak{p}(\underline{\mu})$ has no analytic continuation along any straight path approaching the triple point $\underline{\mu}^*$, contained in the shaded regions of Figure 8.2. The same holds for the two paths approaching the triple point along $\mu_1 = 0$: along $\mu_2 \searrow \mu_2^*$ on the line of coexistence $U_+ \cap U_-$ or along $\mu_2 \nearrow \mu_2^*$ in the pure 0 phase.*

The difference with two phase models is that one must consider, in the construction of the phase diagram as well as in the study of the derivatives, three isoperimetric problems that depend on the chosen path. It is during the study of the solutions of these problems that the restriction on the path appears.

8.2 Open Problems

Our results, which hold for finite range lattice systems at low temperature, give a good picture of how non-analyticity emerges from the first principles of statistical mechanics. Yet, some points still deserve to be studied more precisely, and some other questions, more difficult, remain open.

[2]Originally, this result was obtained for a three phase model of percolation [Fr], along the paths $\mu_2 = \mu_2^*$, $\mu_1 \searrow 0$ and $\mu_1 = 0$, $\mu_2 \nearrow \mu_2^*$.

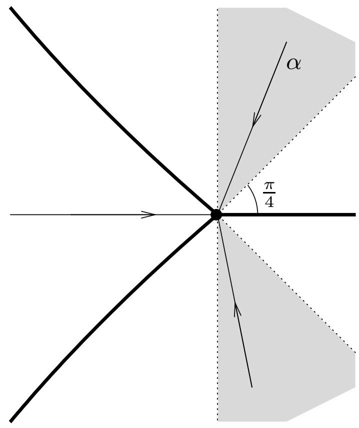

Figure 8.2: Non-Analyticity at the triple point: the pressure has no analytic continuation along any of the paths α approaching the triple point $\underline{\mu}^*$, contained in the shaded region. The same holds for the paths approaching the triple point along $\mu_1 = 0$.

8.2.1 A Different Method

Isakov's technique remains the unique rigorous method for studying non-analyticity. The method imposes to consider two classes of contours. On one hand, the k-large contours are those for which the stationary phase analysis allows to estimate the k-th derivative of the function $u_\Lambda^+(\Gamma)$, with high accuracy. On the other hand, the k-small contours are those for which nothing can be said, even about the sign of the derivative. We then made a particular choice of a k-large contour having a good isoperimetric ratio, and showed that the contribution from the translates of this single contour dominated the contribution from *all* the k-small contours.

This mechanism suggests that the k-th derivative is concentrated on a *class* of contours, but we extracted the contribution from a *single* one (with its translates). A new proof, including a better understanding of this concentration property, would be very valuable, and would allow, hopefully, to give a meaning to the limit

$$\lim_{k\to\infty} \left(\mathfrak{p}^{(k)}(0^+)k!^{-\frac{d}{d-1}}\right)^{-k}. \tag{8.14}$$

This was studied partly by Isakov [Isakov3] for the two dimensional Ising Model; he could relate the limit (8.14) to the asymptotic behaviour of the two point function. Pursuing this analysis, we can hope to relate more accurately the phenomenon of non-analyticity to a basic physical quantity, such as the surface tension (see the review [BIV]).

8.2.2 A General Result for Multiple Phase Systems

The results for models with more than two phases are only partial. As our discussion on the Blume-Capel Model suggests (Section 8.1), it is reasonable to conjecture that there exists a result of the following kind:

In the general framework of Pirogov-Sinai Theory, the pressure is non-analytic along any path crossing a coexistence strata, of arbitrary dimension.

The problem is to obtain uniformity in the temperature. The first step towards such a general theorem is to understand the study of paths, in the Blume-Capel Model, from $\mu_1 > 0$ to $\mu_1 < 0$ with μ_2 fixed, $\mu_2 - \mu_2^* > 0$; the point is to obtain non-analyticity along each of these paths for a range of temperatures which is uniform in $\mu_2 - \mu_2^*$. To this end, one must understand the role played by the unstable phase 0 in the transition $+\backslash-$, arbitrarily close to the triple point.

8.2.3 Analytic Continuation Around the Singularity

In the two phase models we investigated, we have always considered paths along the real axis. A challenging problem which remains is to show whether the analytic continuation can be done along paths that go *around* the singularity. Such continuations were studied by Langer [L] and Fisher [F], in the simple case of the droplet model. The imaginary part of the analytically continued pressure was interpreted, by Langer, as inversely proportional to the lifetime of a metastable state.

For the Ising Model, the problem amounts to showing the existence of analytic continuation along paths such as the one depicted on Figure 8.3. In [NS], Newman and Schulman conjectured that such continuation exists. However, it is not clear, in this case, whether the predictions made by the droplet model are correct.

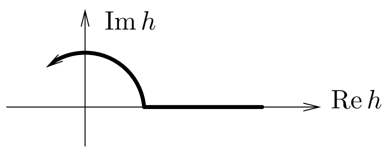

Figure 8.3: An open problem: the analytic continuation in the complex plane, around the singularity $h = 0$ of the Ising Model.

To illustrate the different possible scenarios, let us go back to the model considered in the Introduction, with $d = 2$ for simplicity (see (1.26)):

$$p_D(z) := \sum_{n\geq 1} e^{-\tau\sqrt{n}} z^n \,. \tag{8.15}$$

We saw that the series (8.15) has a radius of convergence equal to one, and that $z = 1$ is a singular point at which derivatives behave like $k!^2$. Since the coefficients of the series are given *explicitly*, exact computations can be done. Indeed, it can be shown that p_D has an analytic continuation to the whole star-shaped domain $\mathbb{C}\backslash[1, +\infty)$, given by

$$\widetilde{p}_D(z) = \int_0^{+\infty} \frac{z}{e^t - z} f(t)\mathrm{d}t \,, \tag{8.16}$$

where

$$f(t) = \frac{1}{2\pi i}\int_{\frac{1}{2}-i\infty}^{\frac{1}{2}+i\infty} e^{-\tau\sqrt{\omega}} e^{\omega t}\mathrm{d}\omega \,. \tag{8.17}$$

That is, $\widetilde{p}_D$ is analytic on $\mathbb{C}\backslash[1, +\infty)$, and coincides with p_D on $\{|z| < 1\}$. The interval $[1, +\infty)$ is a branch cut of $\widetilde{p}_D$.

The droplet singularity at $z = 1$ was the guiding mechanism for the proofs presented in the bulk of this thesis. A reasonable guess is thus to infer that the Ising Model can be continued analytically around $h = 0$. Nevertheless, we will now see that a slight modification of the coefficients of the pressure leads to important changes in the analytic behaviour on the boundary of the disc $\{|z| < 1\}$.

In [Br], Bricmont [3] considered the following series:

$$p_B(z) := \sum_{n\geq 1} e^{-\tau\lfloor\sqrt{n}\rfloor} z^n \,, \tag{8.18}$$

where $\lfloor x\rfloor$ denotes the largest integer smaller than x. This series provides a different reasonable approximation of the pressure of a lattice gas and it also has a singularity at $z = 1$. Nevertheless, as Bricmont noted, the series (8.18) has a lacunary structure, which implies that it has $\{|z| = 1\}$ as a natural boundary [4]: $p_B(z)$ has no analytic continuation outside the disc $\{|z| < 1\}$, across any point $z_0 \in \{|z| = 1\}$. Since (8.15) and (8.18) have very different analytic structures, and since neither of them is a priori better for approximating the free energy in

[3] The same remark was made by Borgs in [Bo].

[4] A series $\sum_n a_n z^{\lambda_n}$ is *lacunary* if $\lim_n \frac{n}{\lambda_n} = 0$. For example $1 + z + z^4 + z^9 + z^{16} + \ldots$. A theorem of Fabry states that any lacunary series has its disc of convergence as a natural boundary (see [Rem2]).

the formation of droplets in a condensing system, it is hard to make a precise guess concerning the continuation of the free energy of the Ising Model around $h = 0$. The first scenario suggests that analytic continuation exists; the second suggests that $\{\mathrm{Re}\, h = 0\}$ is a natural boundary.

Actually, there exists no simple model - other than those of Langer and Fisher - in which this problem can be solved. For example, the exact asymptotic behaviour of the coefficients of the model of Kunz and Souillard (see (1.31)) is known [5], but nothing can be said about the analytic properties on the boundary of the disc of convergence (!). The problem is that the analytic behaviour of a series on the boundary of its disc of convergence is so sensitive to the value of the coefficients [6], that any kind of approximation can be misleading regarding analytic properties. This was emphasized by Penrose [Pe]:

> *Langer's derivation, however, uses the approximation of replacing an infinite series formula for the free energy of an Ising ferromagnet by the corresponding integral. Since analytic continuation is a form of extrapolation, the uncontrolled errors introduced by this approximation might have a profound effect on the analytically continued free energy.*

Concerning more realistic models, our technique does not answer the question, although most of our analysis goes through also when considering purely imaginary magnetic fields (the phase diagram, for instance, is constructed in a neighbourhood of the imaginary axis). Namely, the fact that we have always considered paths $h \searrow 0$ along the real axis has had the crucial consequence, in the final part of the argument giving the lower bound on the k-th derivative of the pressure, that the contribution from each k-large contour, besides being real, had the *same sign*. For purely imaginary magnetic fields, k-large and k-small contours can be defined, but each term of the sum over k-large contours is a complex number, and nothing can be said about the value of the sum.

It can be concluded with relative certainty that the study of this problem requires a different technique, with a better understanding of the properties of the Ising Model in a purely imaginary magnetic field.

[5] In [ACC], the asymptotic behaviour of the cluster size distribution was computed exactly, and related to the Wulff construction.

[6] In [Po], Polyá emphasised the sensivity to the coefficients by comparing the set $\mathcal{S}$ of series that have a radius of convergence equal to one, with the subset $\mathcal{S}' \subset \mathcal{S}$ of series that have $\{|z| = 1\}$ as a natural boundary. By introducing a suitable topology on $\mathcal{S}$, Polyá showed that $\mathcal{S}'$ is open and dense in $\mathcal{S}$.

8.2.4 Long Range Models $|i-j|^{-\alpha}$

Our results hold for any finite long *but finite* range potential, in dimension $d \geq 2$. In fact our technique certainly applies to any system to which the Pirogov-Sinai Theory applies. For instance, with infinite range interactions decaying exponentially fast. Namely, we have seen that the restricted phases, in the study of the Kac model, introduced a long range interaction between the contours, decaying exponentially fast. This interaction was treated via a linking procedure.

We can also expect our results to extend to systems with interactions of the type $|i-j|^{-\alpha}$, with α sufficiently large [7]. A situation which deserves special attention is the one dimensional case, with $\alpha = 2$. It was shown by Fröhlich and Spencer [FS] that this model exhibits a first order phase transition, and an interesting problem would be to study the analytic properties of this model at the transition point. It is not clear whether the droplet mechanism, which we used in dimensions greater or equal to two, can be used to exhibit a non-analytic behaviour for a one dimensional system. The analysis seems more delicate, and might require a completely different approach.

[7] The Pirogov-Sinai Theory of such potentials was developed by Park [Pa].

Appendix A

General Two phases Models

This appendix contains the proof of Theorem 1.2. We have left it in the same form in which it was when submitted for publication. Therefore, notations and terminology differ slightly from the rest of the text, and repetitions might occur at a few places.

A.1 Introduction

We study a lattice model with finite state space on $\mathbf{Z}^d$, $d \geq 2$. Let H_0 be a Hamiltonian with finite-range periodic interaction, having two periodic ground-states ψ_1 and ψ_2, and so that Peierls condition is verified. Let H_1 be a Hamiltonian with periodic and finite range interaction, so that the perturbed Hamiltonian

$$H^\mu = H_0 + \mu H_1$$

splits the degeneracy of the ground-states of H_0: if $\mu < 0$, then H^μ has a unique ground-state ψ_2, and if $\mu > 0$, then H^μ has a unique ground-state ψ_1. The free energy of the model with Hamiltonian H^μ, at inverse temperature β, is denoted by $f(\mu, \beta)$. Our main result is

Theorem A.1. *Under the above setting, there exist an open interval $U_0 \ni 0$, $\beta^* \in \mathbb{R}^+$ and, for all $\beta \geq \beta^*$, $\mu^*(\beta) \in U_0$ with the following properties.*

1. *There is a first-order phase transition at $\mu^*(\beta)$.*
2. *The free energy $f(\mu, \beta)$ is analytic in μ in $\{\mu \in U_0 : \mu < \mu^*(\beta)\}$; it has a C^∞ continuation in $\{\mu \in U_0 : \mu \leq \mu^*(\beta)\}$.*
3. *The free energy $f(\mu, \beta)$ is analytic in μ in $\{\mu \in U_0 : \mu > \mu^*(\beta)\}$; it has a C^∞ continuation in $\{\mu \in U_0 : \mu \geq \mu^*(\beta)\}$.*
4. *There is no analytic continuation of f from $\mu < \mu^*(\beta)$ to $\mu > \mu^*(\beta)$ across $\mu^*(\beta)$, or vice-versa.*

This theorem answers a fundamental theoretical question: does the free energy, which is analytic in the region of a single phase, have an analytic continuation beyond a first-order phase transition point? The answer is yes for the theory of a simple fluid of van der Waals or for mean-field theories. The analytic continuation of the free energy beyond the transition point was interpreted as the free energy of a metastable phase. The answer is no for models with finite range interaction, under very general conditions, as Theorem A.1 shows. This contrasted behavior has its origin in the fact that for models with finite range interaction there is spatial phase separation at first order phase transition, contrary to what happens in a mean-field model. Theorem A.1 and its proof confirm the prediction of the droplet model [F].

Theorem A.1 generalizes the works of Isakov [Isakov1] for the Ising model and [Isakov2], where a similar theorem is proven under additional assumptions, which are not easy to verify in a concrete model. Our version of Theorem A.1, which relies uniquely on Peierls condition, is therefore a genuine improvement of [Isakov2]. The first result of this kind was proven by Kunz and Souillard [KuS]; it concerns the non-analytic behavior of the generating function of the cluster size distribution in percolation, which plays the role of a free energy in that model. The first statement of Theorem A.1 is a particular case of the theory of Pirogov and Sinai (see [S]). We give a proof of this result, as far as it concerns the free energy, since we need detailed informations about the phase diagram in the complex plane of the parameter μ.

The obstruction to an analytic continuation of the free energy in the variable μ is due to the stability of the droplets of both phases in a neighborhood of μ^*. Our proof follows for the essential that of Isakov in [Isakov1]. We give a detailed proof of Theorem A.1, and do not assume any familiarity with [Isakov1] or [Isakov2]. On the other hand we assume that the reader is familiar with the cluster expansion technique.

The results presented here are true for a much larger class of systems, but for the sake of simplicity we restrict our discussion in that paper to the above setting, which is already quite general. For example, Theorem A.1 is true for Potts model with high number q of components at the first order phase transition point β_c, where the q ordered phases coexist with the disordered phase. Here $\mu = \beta$, the inverse temperature, and the statement is that the free energy, which is analytic for $\beta > \beta_c$, or for $\beta < \beta_c$, does not have an analytic continuation across β_c. Theorem A.1 is also true when the model has more than two ground-states. For example, for the Blume-Capel model, whose Hamiltonian is

$$\sum_{i,j}(s_i - s_j)^2 - h\sum_i s_i - \lambda\sum_i s_i^2 \quad \text{with} \quad s_i \in \{-1, 0, 1\},$$

the free energy is an analytic function of h and λ in the single phase regions. At low temperature, at the triple point occurring at $h = 0$ and $\lambda = \lambda^*(\beta)$ there is

no analytic continuation of the free energy in λ, along the path $h = 0$, or in the variable h, along the path $\lambda = \lambda^*$. The case of coexistence of more than two phases will be treated in a separate paper.
In the rest of the section we fix the main notations following chapter two of Sinai's book [S], so that the reader may easily find more information if necessary. We also state Lemma A.1 which contains all estimates on partition functions or free energies. We omit the proof, which relies on the cluster expansion method.
The model is defined on the lattice $\mathbf{Z}^d$, $d \geq 2$. The spin variables $\varphi(x)$, $x \in \mathbf{Z}^d$, take values in a finite state space. If φ, ψ are two spin configurations, then $\varphi = \psi$ (a.s.) means that $\varphi(x) \neq \psi(x)$ holds only on a finite subset of $\mathbf{Z}^d$. The restriction of φ to a subset $A \subset \mathbf{Z}^d$ is denoted by $\varphi(A)$. The cardinality of a subset S is denoted by $|S|$. If $x, y \in \mathbf{Z}^d$, then $|x - y| := \max_{i=1}^d |x_i - y_i|$; if $W \subset \mathbf{Z}^d$ and $x \in \mathbf{Z}^d$, then $d(x, W) := \min_{y \in W} |x - y|$ and if W, W' are subsets of $\mathbf{Z}^d$, then $d(W, W') = \min_{x \in W} d(x, W')$. We define for $W \subset \mathbf{Z}^d$

$$\partial W := \{x \in W : \, d(x, \mathbf{Z}^d \backslash W) = 1\} .$$

A subset $W \subset \mathbf{Z}^d$ is connected if any two points $x, y \in W$ are connected by a path $\{x_0, x_1, \ldots, x_n\} \subset W$, with $x_0 = x$, $x_n = y$ and $|x_i - x_{i+1}| = 1$, $i = 0, 1, \ldots, n-1$. A component is a maximally connected subset.
Let H be a Hamiltonian with finite-range and periodic bounded interaction. By introducing an equivalent model on a sublattice, with a larger state space, we can assume that the model is translation invariant with interaction between neighboring spins $\varphi(x)$ and $\varphi(y)$, $|x - y| = 1$, only. Therefore, without restricting the generality, we assume that this is the case and that the interaction is $\mathbf{Z}^d$-invariant. The Hamiltonian is written

$$H^\mu = H_0 + \mu H_1 \,, \quad \mu \in \mathbb{R} \,.$$

H_0 has two $\mathbf{Z}^d$-invariant ground-states ψ_1 and ψ_2, and the perturbation H_1 splits the degeneracy of the ground-states of H_0. We assume that the energy (per unit spin) of the ground-states of H_0 is 0. $\mathcal{U}_x^\mu(\varphi) \equiv \mathcal{U}_{0,x} + \mu \mathcal{U}_{1,x}$ is the interaction energy of the spin located at x for the configuration φ, so that by definition

$$H^\mu(\varphi) = \sum_{x \in \mathbf{Z}^d} \mathcal{U}_x^\mu(\varphi) \quad \text{(formal sum)} \,.$$

$\mathcal{U}_{1,x}$ is an order parameter for the phase transition. If φ and ψ are two configurations and $\varphi = \psi$ (a.s.), then

$$H^\mu(\varphi|\psi) := \sum_{x \in \mathbf{Z}^d} \big(\mathcal{U}_x^\mu(\varphi) - \mathcal{U}_x^\mu(\psi)\big) \,.$$

This last sum is finite since only finitely many terms are non-zero. The main condition, which we impose on H_0, is Peierls condition for the ground-states ψ_1

and ψ_2. Let $x \in \mathbf{Z}^d$ and

$$W_1(x) := \{y \in \mathbf{Z}^d : |y - x| \leq 1\}.$$

The boundary $\partial\varphi$ of the configuration φ is the subset of $\mathbf{Z}^d$ defined by

$$\partial\varphi := \bigcup_{x \in \mathbf{Z}^d} \big\{W_1(x) : \varphi(W_1(x)) \neq \psi_m(W_1(x)),\ m = 1, 2\big\}.$$

Peierls condition means that there exists a positive constant ρ such that for $m = 1, 2$

$$H_0(\varphi|\psi_m) \geq \rho|\partial\varphi| \quad \forall\ \varphi \text{ such that } \varphi = \psi_m \text{ (a.s.)}.$$

We shall not write usually the μ-dependence of some quantity; we write for example H or $\mathcal{U}_x$ instead of H^μ or $\mathcal{U}_x^\mu$.

Definition A.1. *Let M denote a finite connected subset of $\mathbf{Z}^d$, and let φ be a configuration. Then a couple $\Gamma = (M, \varphi(M))$ is called a contour of φ if M is a component of the boundary $\partial\varphi$ of φ. A couple $\Gamma = (M, \varphi(M))$ of this type is called a contour if there exists at least one configuration φ such that Γ is a contour of φ.*

If $\Gamma = (M, \varphi(M))$ is a contour, then M is the support of Γ, which we also denote by $\operatorname{supp}\Gamma$. Suppose that $\Gamma = (M, \varphi(M))$ is a contour and consider the components A_α of $\mathbf{Z}^d \backslash M$. Then for each component A_α there exists a unique ground-state $\psi_{q(\alpha)}$, such that for each $x \in \partial A_\alpha$ one has $\varphi(W_1(x)) = \psi_{q(\alpha)}(W_1(x))$. The index $q(\alpha)$ is the label of the component A_α. For any contour Γ there exists a unique infinite component of $\mathbf{Z}^d \backslash \operatorname{supp}\Gamma$, $\operatorname{Ext}\Gamma$, called the exterior of Γ; all other components are called internal components of Γ. The ground-state corresponding to the label of $\operatorname{Ext}\Gamma$ is the boundary condition of Γ; the superscript q in Γ^q indicates that Γ is a contour with boundary condition ψ_q. $\operatorname{Int}_m\Gamma$ is the union of all internal components of Γ with label m; $\operatorname{Int}\Gamma := \bigcup_{m=1,2} \operatorname{Int}_m\Gamma$ is the interior of Γ. We use the abbreviations $|\Gamma| := |\operatorname{supp}\Gamma|$ and $V_m(\Gamma) := |\operatorname{Int}_m\Gamma|$. We define[1]

$$V(\Gamma^q) := V_m(\Gamma^q) \quad m \neq q. \tag{A.1}$$

For $x \in \mathbf{Z}^d$, let

$$c(x) := \big\{y \in \mathbb{R}^d : \max_{i=1}^{d} |x_i - y_i| \leq 1/2\big\}$$

be the unit cube of center x in $\mathbb{R}^d$. If $\Lambda \subset \mathbf{Z}^d$, then $|\Lambda|$ is equal to the d-volume of

$$\bigcup_{x \in \Lambda} c(x) \subset \mathbb{R}^d. \tag{A.2}$$

[1]Here our convention differs from [S].

The $(d-1)$-volume of the boundary of the set (A.2) is denoted by $\partial|\Lambda|$. We have

$$2d\,|\Lambda|^{\frac{d-1}{d}} \leq \partial|\Lambda|\,. \tag{A.3}$$

The equality in (A.3) is true for cubes only. When $\Lambda = \mathrm{Int}_m\,\Gamma^q$, $m \neq q$, $V(\Gamma^q) \equiv |\Lambda|$ and $\partial V(\Gamma^q) \equiv \partial|\Lambda|$; there exists a positive constant C_0 such that

$$\partial V(\Gamma^q) \leq C_0|\Gamma^q| \quad q = 1, 2\,. \tag{A.4}$$

For each contour $\Gamma = (M, \varphi(M))$ there corresponds a unique configuration φ_Γ with the properties: $\varphi_\Gamma = \psi_q$ on $\mathrm{Ext}\,\Gamma$, where q is the label of $\mathrm{Ext}\,\Gamma$, $\varphi_\Gamma(M) = \varphi(M)$, $\varphi_\Gamma = \psi_m$ on $\mathrm{Int}_m\,\Gamma$, $m = 1, 2$. Γ is the only contour of φ_Γ. Let $\Lambda \subset \mathbf{Z}^d$; the notation $\Gamma \subset \Lambda$ means that $\mathrm{supp}\,\Gamma \subset \Lambda$, $\mathrm{Int}\,\Gamma \subset \Lambda$ and $d(\mathrm{supp}\,\Gamma, \Lambda^c) > 1$. A contour Γ of a configuration φ is an external contour of φ if and only if $\Gamma \subset \mathrm{Ext}\,\Gamma'$ for any contour Γ' of φ.

Definition A.2. *Let $\Omega(\Gamma^q)$ be the set of configurations $\varphi = \psi_q$ (a.s.) such that Γ^q is the only external contour of φ. Then*

$$\Theta(\Gamma^q) := \sum_{\varphi\in\Omega(\Gamma^q)} \exp\big[-\beta H(\varphi|\psi_q)\big]\,.$$

Let $\Lambda \subset \mathbf{Z}^d$ be a finite subset; let $\Omega_q(\Lambda)$ be the set of configurations $\varphi = \psi_q$ (a.s.) such that $\Gamma \subset \Lambda$ whenever Γ is a contour of φ. Then

$$\Theta_q(\Lambda) := \sum_{\varphi\in\Omega_q(\Lambda)} \exp\big[-\beta H(\varphi|\psi_q)\big]\,.$$

Two fundamental identities relate the partition functions $\Theta(\Gamma^q)$ and $\Theta_q(\Lambda)$.

$$\Theta_q(\Lambda) = \sum \prod_{i=1}^{n} \Theta(\Gamma_i^q)\,, \tag{A.5}$$

where the sum is over the set of all families $\{\Gamma_1^q, \ldots, \Gamma_n^q\}$ of external contours in Λ, and

$$\Theta(\Gamma^q) = \exp\big[-\beta H(\varphi_{\Gamma^q}|\psi_q)\big] \prod_{m=1}^{2} \Theta_m(\mathrm{Int}_m\,\Gamma^q)\,. \tag{A.6}$$

We define (limit in the sense of van Hove)

$$g_q := \lim_{\Lambda\uparrow\mathbf{Z}^d} -\frac{1}{\beta|\Lambda|} \log\Theta_q(\Lambda)\,.$$

The energy (per unit volume) of ψ_m for the Hamiltonian H_1 is

$$h(\psi_m) := \mathcal{U}_{1,x}(\psi_m)\,.$$

By definition of H_1, $h(\psi_2) - h(\psi_1) \neq 0$, and we assume that

$$\Delta := h(\psi_2) - h(\psi_1) > 0\,.$$

The free energy in the thermodynamical limit is

$$f = \lim_{\Lambda\uparrow \mathbf{Z}^d} -\frac{1}{\beta|\Lambda|}\log\Theta_q(\Lambda) + \lim_{\Lambda\uparrow \mathbf{Z}^d}\frac{1}{|\Lambda|}\sum_{x\in\Lambda}\mathcal{U}_x(\psi_q) = g_q + \mu\, h(\psi_q)\,. \tag{A.7}$$

It is independent of the boundary condition ψ_q.

Definition A.3. *Let Γ^q be a contour with boundary condition ψ_q. The weight $\omega(\Gamma^q)$ of Γ^q is*

$$\omega(\Gamma^q) := \exp\big[-\beta H(\varphi_{\Gamma^q}|\psi_q)\big]\prod_{m:m\neq q}\frac{\Theta_m(\mathrm{Int}_m\,\Gamma^q)}{\Theta_q(\mathrm{Int}_m\,\Gamma^q)}\,.$$

The (bare) surface energy of a contour Γ^q is

$$\|\Gamma^q\| := H_0(\varphi_{\Gamma^q}|\psi_q)\,.$$

For a contour Γ^q we set

$$a(\varphi_{\Gamma^q}) := \sum_{x\in\mathrm{supp}\,\Gamma^q}\mathcal{U}_{1,x}(\varphi_{\Gamma^q}) - \mathcal{U}_{1,x}(\psi_q)\,.$$

Since the interaction is bounded, there exists a constant C_1 so that

$$|a(\varphi_{\Gamma^q})| \leq C_1|\Gamma^q|\,. \tag{A.8}$$

Using these notations we have

$$\begin{aligned} H(\varphi_{\Gamma^q}|\psi_q) &= \sum_{x\in\mathrm{supp}\,\Gamma^q}\big(\mathcal{U}_x(\varphi_{\Gamma^q}) - \mathcal{U}_x(\psi_q)\big) + \sum_{x\in\mathrm{Int}\,\Gamma^q}\big(\mathcal{U}_x(\varphi_{\Gamma^q}) - \mathcal{U}_x(\psi_q)\big) \\ &= H_0(\varphi_{\Gamma^q}|\psi_q) + \mu a(\varphi_{\Gamma^q}) + \mu(h(\psi_m) - h(\psi_q))V(\Gamma^q) \\ &= \|\Gamma^q\| + \mu a(\varphi_{\Gamma^q}) + \mu(h(\psi_m) - h(\psi_q))V(\Gamma^q) \quad (m\neq q)\,. \end{aligned} \tag{A.9}$$

The surface energy $\|\Gamma^q\|$ is always strictly positive since Peierls condition holds, and there exists a constant C_2, independent of $q = 1, 2$, such that

$$\rho|\Gamma^q| \leq \|\Gamma^q\| \leq C_2|\Gamma^q|\,. \tag{A.10}$$

Definition A.4. *The weight $\omega(\Gamma^q)$ is τ-stable for Γ^q if*

$$|\omega(\Gamma^q)| \leq \exp(-\tau|\Gamma^q|)\,.$$

For finite subset $\Lambda \subset \mathbf{Z}^d$, using (A.5) and (A.6), one obtains easily the following identity for the partition function $\Theta_q(\Lambda)$,

$$\Theta_q(\Lambda) = 1 + \sum \prod_{i=1}^{n} \omega(\Gamma_i^q)\,, \tag{A.11}$$

where the sum is over all families of compatible contours $\{\Gamma_1^q, \ldots, \Gamma_n^q\}$ with boundary condition ψ_q, that is, $\Gamma_i^q \subset \Lambda$ and $d(\operatorname{supp}\Gamma_i^q, \operatorname{supp}\Gamma_j^q) > 1$ for all $i \neq j$, $i, j = 1, \ldots, n$, $n \geq 1$. If the weights of all contours with boundary condition ψ_q are τ-stable and if τ is large enough, then one can express the logarithm of $\Theta_q(\Lambda)$ as an absolutely convergent sum,

$$\log \Theta_q(\Lambda) = \sum_{m \geq 1} \frac{1}{m!} \sum_{\Gamma_1^q \subset \Lambda} \cdots \sum_{\Gamma_m^q \subset \Lambda} \varphi_m^T(\Gamma_1^q, \ldots, \Gamma_m^q) \prod_{i=1}^{m} \omega(\Gamma_i^q)\,. \tag{A.12}$$

In (A.12) $\varphi_m^T(\Gamma_1^q, \ldots, \Gamma_m^q)$ is a purely combinatorial factor. This is the basic formula which is used for controlling $\Theta_q(\Lambda)$. We also introduce restricted partition functions and free energies. For each $n = 0, 1, \ldots$, we define new weights $\omega_n(\Gamma^q)$

$$\omega_n(\Gamma^q) := \begin{cases} \omega(\Gamma^q) & \text{if } V(\Gamma^q) \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

For $q = 1, 2$, we define Θ_q^n by equation (A.11), replacing $\omega(\Gamma^q)$ by $\omega_n(\Gamma^q)$, and we set (provided that $\Theta_q^n(\Lambda) \neq 0$ for all Λ)

$$g_q^n := -\lim_{\Lambda \uparrow \mathbf{Z}^d} \frac{1}{\beta|\Lambda|} \log \Theta_q^n(\Lambda) \quad \text{and} \quad f_q^n := g_q^n + z\, h(\psi_q)\,. \tag{A.13}$$

f_q^n is the restricted free energy of order n and boundary condition ψ_q. Let

$$l(n) := C_0^{-1} \lceil 2dn^{\frac{d-1}{d}} \rceil \quad n \geq 1\,. \tag{A.14}$$

Notice that $\Theta_q^n(\Lambda) = \Theta_q(\Lambda)$ if $|\Lambda| \leq n$, and that $V(\Gamma^q) \geq n$ implies that $|\Gamma^q| \geq l(n)$ since (A.3) and (A.4) hold.

Lemma A.1. *Suppose that the weights $\omega(\Gamma^q)$ are τ-stable for all Γ^q. Then there exists $K_0 < \infty$ and $\tau_0^* < \infty$, so that for all $\tau \geq \tau_0^*$,* (A.12) *is absolutely convergent and*

$$\beta|g_q| \leq K_0 \mathrm{e}^{-\tau}\,.$$

For all subsets $\Lambda \subset \mathbf{Z}^d$,

$$\bigl|\log \Theta_q(\Lambda) + \beta g_q\,|\Lambda|\bigr| \leq K_0 \mathrm{e}^{-\tau}\,\partial|\Lambda|\,.$$

If $\omega(\Gamma^q)=0$ *for all* Γ^q *such that* $|\Gamma^q|\leq m$*, then*

$$\beta|g_q|\leq \left(K_0 \mathrm{e}^{-\tau}\right)^m .$$

For $n\geq 1$ *and* $m\geq n$*,*

$$\beta|g_q^m - g_q^{n-1}|\leq \left(K_0 \mathrm{e}^{-\tau}\right)^{l(n)} .$$

Furthermore, if $\omega(\Gamma^q)$ *depends on a parameter* t *and*

$$\big|\frac{d}{dt}\omega(\Gamma^q)\big|\leq D_1\mathrm{e}^{-\tau|\Gamma^q|} \quad \textit{and} \quad \big|\frac{d^2}{dt^2}\omega(\Gamma^q)\big|\leq D_2\mathrm{e}^{-\tau|\Gamma^q|},$$

then there exists $K_k<\infty$ *and* $\tau_k^*<\infty$*,* $k=1,2$*, so that for all* $\tau\geq\tau_k^*$*,* $\frac{d^k}{dt^k}g_q$ *exists and*

$$\beta\big|\frac{d}{dt}g_q\big|\leq D_1K_1\mathrm{e}^{-\tau} \quad \textit{and} \quad \beta\big|\frac{d^2}{dt^2}g_q\big|\leq \max\{D_2,D_1^2\}K_2\mathrm{e}^{-\tau} .$$

For all subsets $\Lambda\subset\mathbf{Z}^d$*,*

$$\big|\frac{d}{dt}\log\Theta_q(\Lambda)+\beta\frac{d}{dt}g_q\,|\Lambda|\big|\leq D_1K_1\mathrm{e}^{-\tau}\,\partial|\Lambda|$$

and

$$\big|\frac{d^2}{dt^2}\log\Theta_q(\Lambda)+\beta\frac{d^2}{dt^2}g_q\,|\Lambda|\big|\leq \max\{D_2,D_1^2\}K_2\mathrm{e}^{-\tau}\,\partial|\Lambda| .$$

Lemma A.1 is proved by the cluster expansion method. It follows from (A.12) and arguments similar to those of the proof of Lemma 3.5. in section 3.3 in [Pf]. The proof of Theorem A.1 is given in the next five subsections. In subsection A.2 we construct the phase diagram and in subsection A.3 we study the analytic continuation of the weights of contours in a neighborhood of the point of phase coexistence μ^*. The results about the analytic continuation are crucial for the rest of the analysis. Construction of the phase diagram in the complex plane has been done by Isakov [Isakov2]. We follow partly this reference and Zahradnik [Z]. In subsection A.4 we derive an expression of the derivatives of the free energy at finite volume. We prove a lower bound for a restricted class of terms of this expression. This is an improved version of a similar analysis of Isakov [Isakov1]. From these results we obtain a lower bound for the derivatives of the free energy f_Λ in a finite box Λ. We show in subsection A.4.1 that for large β, there exists an increasing diverging sequence $\{k_n\}$, so that the k_n^{th}-derivative of f_Λ with respect to μ, evaluated at μ^*, behaves as $k_n!^{\frac{d}{d-1}}$ (provided that Λ is large enough). In the last subsection we end the proof of the impossibility of an analytic continuation of the free energy across μ^*, by showing that the results of subsection A.4.1 remain true in the thermodynamical limit.

A.2 Construction of the Phase Diagram

We construct the phase diagram for complex values of the parameter μ, by constructing iteratively the phase diagram for the restricted free energies f_q^n (see (A.13)). We set $z := \mu + i\nu$. The method consists in finding a sequence of intervals for each $\nu \in \mathbb{R}$,

$$U_n(\nu;\beta) := (\mu_n^*(\nu;\beta) - b_n^1, \mu_n^*(\nu;\beta) + b_n^2)\,,$$

with the properties

$$(\mu_n^*(\nu;\beta) - b_n^1, \mu_n^*(\nu;\beta) + b_n^2) \subset (\mu_{n-1}^*(\nu;\beta) - b_{n-1}^1, \mu_{n-1}^*(\nu;\beta) + b_{n-1}^2) \quad \text{(A.15)}$$

and $\lim_n b_n^q = 0$, $q = 1, 2$. By construction of the intervals $U_{n-1}(\nu;\beta)$ the restricted free energies f_q^{n-1} of order $n-1$, $q = 1, 2$, are well-defined and analytic on

$$\mathbb{U}_{n-1} := \{z \in \mathbb{C} : \mathrm{Re} z \in U_{n-1}(\mathrm{Im} z;\beta)\}\,.$$

The point $\mu_n^*(\nu;\beta)$, $n \geq 1$, is solution of the equation

$$\mathrm{Re}\big(f_2^{n-1}(\mu_n^*(\nu;\beta) + i\nu) - f_1^{n-1}(\mu_n^*(\nu;\beta) + i\nu)\big) = 0\,.$$

$\mu_n^*(0;\beta)$ is the point of phase coexistence for the restricted free energies of order $n-1$, and the point of phase coexistence of the model is given by $\mu^*(0;\beta) = \lim_n \mu_n^*(0;\beta)$. This iterative procedure also gives the necessary results needed in subsection A.2 about the analytic continuation of the weights $\omega(\Gamma^q)$ around the point of phase coexistence μ^*. Since we need sharp results about the analytic continuation of the weights $\omega(\Gamma^q)$, we must choose carefully the two sequences $\{b_n^q\}$, $q = 1, 2$. In order to ease the exposition we first describe the iterative procedure with a specific choice of $\{b_n^q\}$, based on the isoperimetric inequality

$$V(\Gamma^q)^{\frac{d-1}{d}} \leq \chi^{-1}\|\Gamma^q\| \quad \forall\, \Gamma^q\,,\ q = 1, 2\,. \quad \text{(A.16)}$$

Existence of χ in (A.16) follows from (A.3), (A.4) and (A.10). Then, in subsection A.3, we make another choice for $\{b_n^q\}$. This iterative construction is given in details in the proof of the Proposition A.1, which is the main result of subsection A.2.

Proposition A.1. *Let* $0 < \varepsilon < \rho$ *and* $0 < \delta < 1$ *so that* $\Delta - 2\delta > 0$. *Set*

$$U_0 := (-C_1^{-1}\varepsilon, C_1^{-1}\varepsilon) \quad \textit{and} \quad \mathbb{U}_0 := \{z \in \mathbb{C} : \mathrm{Re} z \in U_0\}$$

and

$$\tau(\beta) := \beta(\rho - \varepsilon) - 3C_0\delta\,.$$

There exists $\beta_0 \in \mathbb{R}^+$ *such that for all* $\beta \geq \beta_0$ *the following holds.*
A) There exists a continuous real-valued function on $\mathbb{R}$, $\nu \mapsto \mu^*(\nu;\beta)$, *so that*

$\mu^*(\nu;\beta)+i\nu\in\mathbb{U}_0$.
B) If $\mu+i\nu\in\mathbb{U}_0$ and $\mu\le\mu^*(\nu;\beta)$, then the weight $\omega(\Gamma^2)$ is $\tau(\beta)$-stable for all contours Γ^2 with boundary condition ψ_2, and analytic in $z=\mu+i\nu$ if $\mu<\mu^*(\nu;\beta)$.
C) If $\mu+i\nu\in\mathbb{U}_0$ and $\mu\ge\mu^*(\nu;\beta)$, then the weight $\omega(\Gamma^1)$ is $\tau(\beta)$-stable for all contours Γ^1 with boundary condition ψ_1, and analytic in $z=\mu+i\nu$ if $\mu>\mu^*(\nu;\beta)$.

Remark A.1. ρ is the constant of the Peierls condition and $\Delta=h(\psi_2)-h(\psi_1)>0$. We may choose δ in such a way that $\delta=\delta(\beta)$ and $\lim_{\beta\to\infty}\delta(\beta)=0$, without changing the theorem. Indeed, the only condition which we need to satisfy is (A.20). So, whenever we need it, we consider δ as function of β, so that by taking β large enough, we have δ as small as we wish.

Proof. The iterative method depends on a free parameter θ', $0<\theta'<1$. On the interval $U_0(\nu;\beta):=(-b_0,b_0)$ with $b_0=\varepsilon C_1^{-1}$, $f_q^0(\mu+i\nu)$ is defined and we set $\mu_0^*(\nu;\beta):=0$. The two decreasing sequences $\{b_n^q\}$, $q=1,2$ and $n\ge 1$, are defined in (A.22). The iterative construction is possible whenever the sequences $\{b_n^q\}$, $q=1,2$, verify (A.21), (A.27) and (A.28). We prove iteratively the following statements.

A. $f_q^n(\mu+i\nu;\beta)$ is defined for all $\mu\in U_{n-1}(\nu;\beta)$, and $\nu\mapsto\mu_n^*(\nu;\beta)$ is a continuous solution of the equation

$$\mathrm{Re}\big(f_2^{n-1}(\mu_n^*(\nu;\beta)+i\nu)-f_1^{n-1}(\mu_n^*(\nu;\beta)+i\nu)\big)=0\,,$$

so that (A.15) holds.

B. On $\mathbb{U}_n$, $\omega_n(\Gamma^q)$ is analytic for any contour Γ^q, $q=1,2$, and $\omega_n(\Gamma^q)$ is $\tau_1(\beta)$-stable (see (A.17)).

C. On $\mathbb{U}_n$, $\big|\frac{d}{dz}\omega_n(\Gamma^q)\big|\le\beta C_3\mathrm{e}^{-\tau_2(\beta)|\Gamma^q|}$ (see (A.18) and (A.19)).

D. For each $n\ge 1$, if $\mu\le\mu_n^*(\nu;\beta)-b_n^1$, then $\omega(\Gamma^2)$ is $\tau(\beta)$-stable for any Γ^2 with boundary condition ψ_2. Similarly, for each $n\ge1$, if $\mu\ge\mu_n^*(\nu;\beta)+b_n^2$, then $\omega(\Gamma^1)$ is $\tau(\beta)$-stable for any Γ^1 with boundary condition ψ_1.

From these results the proposition follows with

$$\mu^*(\nu;\beta)=\lim_{n\to\infty}\mu_n^*(\nu;\beta)\,.$$

The analyticity of the weights $\omega(\Gamma^q)$ is an immediate consequence of their stability since $\Theta_m(\mathrm{Int}_m\,\Gamma^q)$ and $\Theta_q(\mathrm{Int}_m\,\Gamma^q)\neq 0$ are analytic.
Let $0<\theta'<1$ be given, as well as ε and δ as in the proposition. We introduce all constants used in the proof below.

$$\tau_1(\beta;\theta'):=\beta\big(\rho(1-\theta')-\varepsilon\big)-2\delta C_0\,, \tag{A.17}$$

$$\tau_2(\beta;\theta') := \tau_1(\beta;\theta') - \frac{d}{d-1}\,, \tag{A.18}$$

and

$$C_3 := C_1 + 2\delta C_0 + (\Delta+2\delta)(\chi^{-1}C_2)^{\frac{d}{d-1}}\,. \tag{A.19}$$

We assume that β_0 is large enough so that[2] $\tau_2(\beta) > \max\{\tau_0^*, \tau_1^*, \tau_2^*\}$, (A.32) holds,

$$K\mathrm{e}^{-\tau_1(\beta)} \leq \delta \quad \text{and} \quad C_3 K\mathrm{e}^{-\tau_2(\beta)} \leq \delta\,, \tag{A.20}$$

where $K = \max\{K_0, K_1\}$, and K_0, K_1 are the constants of Lemma A.1. We assume that for $q = 1, 2$,

$$b_n^q - b_{n+1}^q > \frac{2\delta^{l(n)}}{\beta(\Delta-2\delta)}\,, \quad \forall n \geq 1\,. \tag{A.21}$$

If we define

$$b_n^1 \equiv b_n^2 := \frac{\chi\theta'}{(\Delta+2\delta)n^{\frac{1}{d}}}\,,\ n \geq 1\,, \tag{A.22}$$

then it is immediate to verify (A.21) when β is large enough or δ small enough. On $\mathbb{U}_0$ all contours Γ with empty interior are $\beta(\rho-\varepsilon)$-stable (see (A.8)), and

$$\Big|\frac{d}{dz}\omega(\Gamma)\Big| \leq \beta C_1|\Gamma|\mathrm{e}^{-\beta(\rho-\varepsilon)|\Gamma|} \leq \beta C_1 \mathrm{e}^{-[\beta(\rho-\varepsilon)-1]|\Gamma|} \leq \beta C_3 \mathrm{e}^{-\tau_2(\beta)|\Gamma|}\,.$$

Assume that the construction has been done for all $m \leq n-1$. By Lemma A.1, if $z \in \mathbb{U}_{n-1}$, then

$$\Big|\frac{d}{dz}g_q^m\Big| \leq C_3 K\mathrm{e}^{-\tau_2(\beta)} \leq \delta \quad m \leq n-1\,. \tag{A.23}$$

A. We prove the existence of $\mu_n^*(\nu;\beta) \in \mathbb{U}_{n-1}$. $\mu_n^*(\nu;\beta)$ is solution of the equation

$$\mathrm{Re}\big(f_2^{n-1}(\mu_n^*(\nu;\beta)+i\nu) - f_1^{n-1}(\mu_n^*(\nu;\beta)+i\nu)\big) = 0\,.$$

Let $F^m(z) := f_2^m(z) - f_1^m(z)$. Then, for $\mu'+i\nu \in \mathbb{U}_{n-1}$,

$$\begin{aligned} F^{n-1}(\mu'+i\nu) &= F^{n-1}(\mu'+i\nu) - F^{n-2}(\mu_{n-1}^*+i\nu) \\ &= F^{n-1}(\mu'+i\nu) - F^{n-1}(\mu_{n-1}^*+i\nu) + F^{n-1}(\mu_{n-1}^*+i\nu) \\ &\quad - F^{n-2}(\mu_{n-1}^*+i\nu) \\ &= \int_{\mu_{n-1}^*}^{\mu'} \frac{d}{d\mu}F^{n-1}(\mu+i\nu)\,d\mu + \big(g_2^{n-1}-g_2^{n-2}\big)(\mu_{n-1}^*+i\nu) \\ &\quad - \big(g_1^{n-1}-g_1^{n-2}\big)(\mu_{n-1}^*+i\nu)\,. \end{aligned} \tag{A.24}$$

[2] τ_k^*, $k = 0, 1, 2$, are defined in Lemma A.1. Condition $\tau_2(\beta) > \tau_2^*$ is needed only in Lemma A.3. We have stated Lemma A.3 separately in order to simplify the proof of Proposition A.1.

If $V(\Gamma)=n-1$, then $|\Gamma|\geq l(n-1)$. Therefore, by Lemma A.1,

$$|\big(g_q^{n-1}-g_q^{n-2}\big)(\mu_{n-1}^*+i\nu)|\leq\beta^{-1}\delta^{l(n-1)}\,. \tag{A.25}$$

If $z'=\mu'+i\nu\in\mathbb{U}_{n-1}$, then (A.24), (A.23) and (A.25) imply

$$\begin{aligned}\Delta(\mu'-\mu_{n-1}^*)+2\delta|\mu'-\mu_{n-1}^*|+2\beta^{-1}\delta^{l(n-1)}&\geq \mathrm{Re}F^{n-1}(z')\\ &\geq\Delta(\mu'-\mu_{n-1}^*)-2\delta|\mu'-\mu_{n-1}^*|-2\beta^{-1}\delta^{l(n-1)}\,.\end{aligned}$$

(A.21) implies

$$b_{n-1}^q>b_{n-1}^q-b_n^q>\frac{2\delta^{l(n-1)}}{\beta(\Delta-2\delta)}\,,$$

so that $\mathrm{Re}F^{n-1}(\mu_{n-1}^*-b_{n-1}^1+i\nu)<0$ and $\mathrm{Re}F^{n-1}(\mu_{n-1}^*+b_{n-1}^2+i\nu)>0$. This proves the existence of μ_n^* and its uniqueness, since $\mu\mapsto\mathrm{Re}F^{n-1}(\mu+i\nu)$ is strictly increasing. Moreover, by putting $\mu'=\mu_n^*(\nu;\beta)$ in (A.24), we get

$$|\mu_n^*(\nu;\beta)-\mu_{n-1}^*(\nu;\beta)|\leq\frac{2\delta^{l(n-1)}}{\beta(\Delta-2\delta)}\,.$$

Therefore $\mathbb{U}_n\subset\mathbb{U}_{n-1}$. The implicit function theorem implies that $\nu\mapsto\mu_n^*(\nu;\beta)$ is continuous (even C^∞).

B. We prove that $\omega_n(\Gamma^q)$ is τ_1-stable for all contours Γ^q, $q=1,2$. Let Γ^q be a contour with $V(\Gamma^q)=n$. All contours contributing to $\Theta_m(\mathrm{Int}_m\,\Gamma^q)$ and $\Theta_q(\mathrm{Int}_m\,\Gamma^q)$ have volumes smaller than $n-1$, so that for these contours $\omega(\Gamma)=\omega_{n-1}(\Gamma)$. If $z\in\mathbb{U}_{n-1}$ (use (A.8), (A.4) and the definition of U_0), then

$$\begin{aligned}|\omega(\Gamma^q)|&=\exp\big[-\beta\mathrm{Re}H(\varphi_{\Gamma^q}|\psi_q)\big]\Big|\prod_{m:m\neq q}\frac{\Theta_m(\mathrm{Int}_m\,\Gamma^q)}{\Theta_q(\mathrm{Int}_m\,\Gamma^q)}\Big| \qquad\qquad (\mathrm{A}.26)\\ &\leq\exp\Big[-\beta\|\Gamma^q\|+\big(\beta\varepsilon+2C_0\delta\big)|\Gamma^q|-\beta\mathrm{Re}\big(f_m^{n-1}-f_q^{n-1}\big)V(\Gamma^q)\Big]\\ &=\exp\Big[-\beta\|\Gamma^q\|+\big(\beta\varepsilon+2C_0\delta\big)|\Gamma^q|-\beta\mathrm{Re}\big(f_m^{n-1}-f_q^{n-1}\big)\frac{V(\Gamma^q)}{\|\Gamma^q\|}\|\Gamma^q\|\Big]\,.\end{aligned}$$

Let $\mu\in U_{n-1}(\nu;\beta)$. We prove that b_n^q verify the following conditions, which imply the τ_1-stability.

$$-\mathrm{Re}\big(f_1^{n-1}-f_2^{n-1}\big)\frac{V(\Gamma^2)}{\|\Gamma^2\|}\leq\theta'\quad\text{if }\mu\leq\mu_n^*+b_n^2\text{ and }V(\Gamma^2)=n, \tag{A.27}$$

$$-\mathrm{Re}\big(f_2^{n-1}-f_1^{n-1}\big)\frac{V(\Gamma^1)}{\|\Gamma^1\|}\leq\theta'\quad\text{if }\mu\geq\mu_n^*-b_n^1\text{ and }V(\Gamma^1)=n. \tag{A.28}$$

For the present choice of $\{b_n^q\}$, the isoperimetric inequality (A.16) implies

$$\frac{V(\Gamma^q)}{\|\Gamma^q\|}\leq\frac{V(\Gamma^q)^{\frac{1}{d}}}{\chi}\quad\forall\,q=1,2\,,$$

and therefore

$$\left|\mathrm{Re}\big(f_m^{n-1}-f_q^{n-1}\big)\right|\frac{V(\Gamma^q)}{\|\Gamma^q\|}=\left|\mathrm{Re}\int_{\mu_n^*}^{\mu}\frac{d}{d\mu}\big(f_m^{n-1}-f_q^{n-1}\big)\,d\mu\right|\frac{V(\Gamma^q)}{\|\Gamma^q\|}$$
$$\le|\mu-\mu_n^*|(\Delta+2\delta)\frac{V(\Gamma^q)}{\|\Gamma^q\|}\le\theta'\,.$$

Conditions (A.27) and (A.28) ensure that on $\mathbb{U}_n$

$$|\omega(\Gamma^q)|\le\exp\Big[-\beta\big(\rho(1-\theta')-\varepsilon-2\beta^{-1}C_0\delta\big)|\Gamma^q|\Big]\,.$$

C. We prove that on $\mathbb{U}_n$

$$\Big|\frac{d}{dz}\omega_n(\Gamma)\Big|\le\beta C_3\mathrm{e}^{-\tau_2(\beta)|\Gamma|}\,.$$

Let $V(\Gamma^q)=n$; from (A.9)

$$\frac{d}{dz}\omega_n(\Gamma^q)=\omega_n(\Gamma^q)\Big(-\beta a(\varphi_{\Gamma^q})-\beta\big(h(\psi_m)-h(\psi_q)\big)V(\Gamma^q)$$
$$+\frac{d}{dz}\big(\log\Theta_m(\mathrm{Int}_m\,\Gamma^q)-\log\Theta_q(\mathrm{Int}_m\,\Gamma^q)\big)\Big)\,.$$

By Lemma A.1 and the isoperimetric inequality (A.3) we get

$$\Big|\frac{d}{dz}\omega_n(\Gamma^q)\Big|\le\beta|\omega_n(\Gamma^q)|\big(|\Gamma^q|(C_1+2\delta C_0)+V(\Gamma^q)(\Delta+2\delta)\big)\tag{A.29}$$
$$\le\beta C_3|\omega_n(\Gamma^q)||\Gamma^q|^{\frac{d}{d-1}}$$
$$\le\beta C_3\mathrm{e}^{-\tau_2(\beta)|\Gamma^q|}\,.$$

D. We prove that $\omega(\Gamma^2)(z)$ is $\tau(\beta)$-stable for any contour Γ^2 with boundary condition ψ_2, if $\mu\le\mu_n^*(\nu;\beta)-b_n^1$. Using the induction hypothesis it is sufficient to prove this statement for $z=\mu+i\nu\in\mathbb{U}_{n-1}$ and $\mu\le\mu_n^*(\nu;\beta)-b_n^1$. If $z=\mu+i\nu\in\mathbb{U}_{n-1}$, then all contours with volume $V(\Gamma)\le n-1$ are $\tau_1(\beta)$-stable, and for $\mu\le\mu_n^*$, $\mu\mapsto\mathrm{Re}(f_1^{n-1}-f_2^{n-1})(\mu+i\nu)$ is strictly decreasing. If $\mu\le\mu_n^*(\nu;\beta)-b_n^1$, then (see (A.22) and (A.21))

$$\beta\mathrm{Re}(f_1^{n-1}-f_2^{n-1})(\mu+i\nu)=-\beta\int_{\mu}^{\mu_n^*}\frac{d}{d\mu}\mathrm{Re}(f_1^{n-1}-f_2^{n-1})(\mu+i\nu)\,d\mu$$
$$\ge-\beta\int_{\mu_n^*-b_n^1}^{\mu_n^*}\frac{d}{d\mu}\mathrm{Re}(f_1^{n-1}-f_2^{n-1})(\mu+i\nu)\,d\mu$$
$$\ge\beta b_n^1(\Delta-2\delta)\ge2\delta^{l(n)}\,.\tag{A.30}$$

First suppose that $V(\Gamma^2) \leq n$. From (A.30) and (A.26) it follows that $\omega(\Gamma^2)$ is $\beta(\rho - \varepsilon - 2\beta^{-1}C_0\delta)$-stable, in particular $\tau(\beta)$-stable. Moreover, if $|\Lambda| \leq n$, then

$$\Big|\exp\big[-\beta z(h(\psi_1) - h(\psi_2))|\Lambda|\big]\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| \leq \mathrm{e}^{3\delta\partial|\Lambda|}\,. \tag{A.31}$$

Indeed, all contours inside Λ are $\tau_1(\beta)$-stable. By Lemma A.1

$$\begin{aligned}\Big|\mathrm{e}^{-\beta z(h(\psi_1)-h(\psi_2))|\Lambda|}\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| &\leq \big|\mathrm{e}^{-\beta(zh(\psi_1)-zh(\psi_2)+g_1^{n-1}-g_2^{n-1})|\Lambda|}\big|\,\mathrm{e}^{2\delta\partial|\Lambda|}\\ &= \mathrm{e}^{-\beta\mathrm{Re}(f_1^{n-1}(z)-f_2^{n-1}(z))|\Lambda|}\mathrm{e}^{2\delta\partial|\Lambda|}\\ &\leq \mathrm{e}^{2\delta\partial|\Lambda|}\,.\end{aligned}$$

To prove point D, we prove by induction on $|\Lambda|$ that (A.31) holds for any Λ. Indeed, if (A.31) is true and if we set $\Lambda := \mathrm{Int}_1\Gamma^2$, then it follows easily from the definition of $\omega(\Gamma^2)$ and from (A.9) that $\omega(\Gamma^2)$ is $\tau(\beta)$-stable.
The argument to prove (A.31) is due to Zahradnik [Z]. The statement is true for $|\Lambda| \leq n$. Suppose that it is true for $|\Lambda| \leq m$, $m > n$, and let $|\Lambda| = m+1$. The induction hypothesis implies that $\omega(\Gamma^2)(z)$ is $\tau(\beta)$-stable if $V(\Gamma^2) \leq m$. Therefore

$$\Big|\mathrm{e}^{-\beta z(h(\psi_1)-h(\psi_2))|\Lambda|}\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| \leq \big|\mathrm{e}^{-\beta(zh(\psi_1)-zh(\psi_2)-g_2^{m})|\Lambda|}\Theta_1(\Lambda)\big|\mathrm{e}^{\delta\partial|\Lambda|}\,.$$

From (A.5)

$$\Theta_1(\Lambda) = \sum\prod_{j=1}^{r}\Theta(\Gamma_j^1)\,,$$

where the sum is over all families $\{\Gamma_1^1,\ldots,\Gamma_r^1\}$ of compatible external contours in Λ. We say that an external contour Γ_j^1 is large if $V(\Gamma_j^1) \geq n$. Suppose that the contours $\Gamma_1^1,\ldots\Gamma_p^1$ are large and all other contours $\Gamma_{p+1}^1,\ldots\Gamma_r^1$ not large. We set

$$\mathrm{Ext}_1^p(\Lambda) := \Big(\bigcap_{j=1}^{p}\mathrm{Ext}\Gamma_j^1\Big)\cap\Lambda\,.$$

Summing over all contours which are not large, and using (A.6), we get

$$\begin{aligned}\Theta_1(\Lambda) &= \sum\Theta_1^{n-1}\big(\mathrm{Ext}_1^p(\Lambda)\big)\prod_{j=1}^{p}\exp\big[-\beta H(\varphi_{\Gamma_j^1}|\psi_1)\big]\Theta_1(\mathrm{Int}_1\Gamma_j^1)\Theta_2(\mathrm{Int}_2\Gamma_j^1)\\ &= \sum\Theta_1^{n-1}\big(\mathrm{Ext}_1^p(\Lambda)\big)\prod_{j=1}^{p}\mathrm{e}^{-\beta\|\Gamma_j^1\|-\beta z a(\varphi_{\Gamma_j^1})+\beta z(h(\psi_1)-h(\psi_2))|\mathrm{Int}_2\Gamma_j^1|}\\ &\quad\cdot\frac{\Theta_1(\mathrm{Int}_1\Gamma_j^1)}{\Theta_2(\mathrm{Int}_1\Gamma_j^1)}\,\Theta_2(\mathrm{Int}_1\Gamma_j^1)\Theta_2(\mathrm{Int}_2\Gamma_j^1)\,;\end{aligned}$$

the sums are over all families $\{\Gamma_1^1, \dots \Gamma_p^1\}$ of compatible external large contours in Λ. All contours which are not large are $\tau_1(\beta)$-stable, and we use the cluster expansion to control $\Theta_1^{n-1}\big(\mathrm{Ext}_1^p(\Lambda)\big)$, $\Theta_2(\mathrm{Int}_1\Gamma_j^1)$ and $\Theta_2(\mathrm{Int}_2\Gamma_j^1)$. Notice that $\partial|\mathrm{Ext}_1^p(\Lambda)| \leq \partial|\Lambda| + \sum_{j=1}^p C_0|\Gamma_j^1|$. By Lemma A.1 and the induction hypothesis,

$$\Big|\mathrm{e}^{-\beta z(h(\psi_1)-h(\psi_2))|\Lambda|}\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| \leq \mathrm{e}^{2\delta\partial|\Lambda|}\sum \mathrm{e}^{-\beta\mathrm{Re}(f_1^{n-1}-f_2^{n-1}-g_2^m+g_2^{n-1})|\mathrm{Ext}_1^p(\Lambda)|} \cdot\prod_{j=1}^p \mathrm{e}^{-(\beta\rho-\beta\varepsilon-6C_0\delta)|\Gamma_j^1|}\mathrm{e}^{-\beta\mathrm{Re}(f_1^{n-1}-f_2^{n-1}-g_2^m+g_2^{n-1})|\Gamma_j^1|}\,.$$

We define

$$\hat{\tau}(\beta) := \beta(\rho-\varepsilon) - 6C_0\delta\,.$$

From (A.30) and Lemma A.1 we have

$$\beta(f_1^{n-1}-f_2^{n-1}-g_2^m+g_2^{n-1}) \geq \delta^{l(n)}\,.$$

Hence,

$$\Big|\mathrm{e}^{-\beta z(h(\psi_1)-h(\psi_2))|\Lambda|}\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| \leq \mathrm{e}^{2\delta\partial|\Lambda|}\sum \mathrm{e}^{-\delta^{l(n)}|\mathrm{Ext}_1^p(\Lambda)|}\prod_{j=1}^p \mathrm{e}^{-(\delta^{l(n)}+\hat{\tau}(\beta))|\Gamma_j^1|}\,.$$

We define

$$\hat{\omega}(\Gamma) := \begin{cases} \mathrm{e}^{-(\hat{\tau}(\beta)-C_0\delta)|\Gamma|} & \text{if } |\Gamma| \geq l(n); \\ 0 & \text{otherwise.} \end{cases}$$

Let $\hat{\Theta}(\Lambda)$ be defined by (A.11), replacing $\omega(\Gamma^q)$ by $\hat{\omega}(\Gamma)$, and let

$$\hat{g} := \lim_{\Lambda\uparrow\mathbf{Z}^d} -\frac{1}{\beta|\Lambda|}\log\hat{\Theta}(\Lambda)\,.$$

We assume that β_0 is large enough so that for all $\beta \geq \beta_0$,

$$K\mathrm{e}^{-\hat{\tau}(\beta)} \leq \delta\,, \tag{A.32}$$

where K is the constant of Lemma A.1. Since $\beta|\hat{g}| \leq \delta^{l(n)}$, putting into evidence a factor $\mathrm{e}^{\beta\hat{g}|\Lambda|}$, we get

$$\begin{aligned}\Big|\mathrm{e}^{-\beta z(h(\psi_1)-h(\psi_2))|\Lambda|}\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| &\leq \mathrm{e}^{2\delta\partial|\Lambda|+\beta\hat{g}|\Lambda|}\sum\prod_{j=1}^p \mathrm{e}^{-\hat{\tau}(\beta)|\Gamma_j^1|}\mathrm{e}^{-\beta\hat{g}|\mathrm{Int}\,\Gamma_j^1|} \\ &\leq \mathrm{e}^{2\delta\partial|\Lambda|+\beta\hat{g}|\Lambda|}\sum\prod_{j=1}^p \mathrm{e}^{-(\hat{\tau}(\beta)-C_0\delta)|\Gamma_j^1|}\hat{\Theta}(\mathrm{Int}\,\Gamma_j^1)\,.\end{aligned} \tag{A.33}$$

In the last line of (A.33) we interpret $e^{-\beta\hat{g}|\mathrm{Int}\,\Gamma^1|}$ as a partition function (up to a boundary term), since by Lemma A.1,

$$e^{-\beta\hat{g}|\mathrm{Int}\,\Gamma^1|} \leq \hat{\Theta}(\mathrm{Int}\,\Gamma^1)\, e^{C_0\delta|\Gamma^1|}\,.$$

We sum over external contours in (A.33) and get

$$\Big| e^{-\beta z(h(\psi_1)-h(\psi_2))|\Lambda|}\frac{\Theta_1(\Lambda)}{\Theta_2(\Lambda)}\Big| \leq e^{2\delta\partial|\Lambda|+\beta\hat{g}|\Lambda|}\hat{\Theta}(\Lambda) \leq e^{3\delta\partial|\Lambda|}\,.$$

□

It is not difficult to prove more regularity for the curve $\nu \mapsto \mu^*(\nu;\beta)$. We need below only the following result.

Lemma A.2. *Let $0 < \delta < 1$. If β is sufficiently large, then for all $n \geq 1$ $\frac{d}{d\nu}\mu_n^*(0;\beta) = 0$, and*

$$\Big|\frac{d^2}{d\nu^2}\mu_n^*(\nu;\beta)\Big| \leq \frac{2\delta}{\Delta - 2\delta}\Big(\Big(\frac{2\delta}{\Delta - 2\delta}\Big)^2 + \frac{2\delta}{\Delta - 2\delta} + 1\Big)\,.$$

Proof. Let δ be as in the proof of Proposition A.1. Because the free energies f_1^{n-1} and f_2^{n-1} are real on the real axis, it follows that $\nu \mapsto \mu_n^*(\nu;\beta)$ is even, and therefore $\frac{d}{d\nu}\mu_n^*(0;\beta) = 0$. By definition $\mu_n^*(\nu;\beta)$ is solution of

$$\mathrm{Re}\big(f_2^{n-1}(\mu_n^*(\nu;\beta)+i\nu) - f_1^{n-1}(\mu_n^*(\nu;\beta)+i\nu)\big) = 0\,,$$

which implies that

$$\Delta\frac{d\mu_n^*}{d\nu} = \frac{d}{d\mu}\mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)\frac{d\mu_n^*}{d\nu} + \frac{d}{d\nu}\mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)$$

and

$$\begin{aligned}\Delta\frac{d^2\mu_n^*}{d\nu^2} &= \frac{d}{d\mu}\mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)\frac{d^2\mu_n^*}{d\nu^2} + \frac{d^2}{d\mu^2}\mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)\big(\frac{d\mu_n^*}{d\nu}\big)^2 \\ &+ \frac{d^2}{d\mu d\nu}\mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)\frac{d\mu_n^*}{d\nu} + \frac{d^2}{d\nu^2}\mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)\,.\end{aligned}$$

From the proof of Proposition A.1 we have on $\mathbb{U}_m$,

$$\Big|\frac{d}{dz}\omega_m(\Gamma)\Big| \leq \beta C_3 e^{-\tau_2(\beta)|\Gamma|}\,.$$

Let $\tau_3(\beta) := \tau_1(\beta) - 2\frac{d}{d-1}$. A similar proof shows that for β sufficiently large, there exists C_4 so that for any m

$$\Big|\frac{d^2}{dz^2}\omega_m(\Gamma)\Big| \leq \beta^2 C_4 e^{-\tau_3(\beta)|\Gamma|}\,.$$

Assume that β is large enough so that

$$\beta \max\{C_4, C_3^2\} K_2 \mathrm{e}^{-\tau_3(\beta)|\Gamma|} \leq \delta \,.$$

Then Lemma A.1 gives for $G^{n-1} := \mathrm{Re}\big(g_1^{n-1} - g_2^{n-1}\big)$

$$\big|\frac{d}{d\mu} G^{n-1}\big| \leq 2\delta \;,\; \big|\frac{d}{d\nu} G^{n-1}\big| \leq 2\delta \,,$$

$$\big|\frac{d^2}{d\mu^2} G^{n-1}\big| \leq 2\delta \;,\; \big|\frac{d^2}{d\nu^2} G^{n-1}\big| \leq 2\delta \;,\; \big|\frac{d^2}{d\mu d\nu} G^{n-1}\big| \leq 2\delta \,.$$

Hence

$$\big|\frac{d\mu_n^*}{d\nu}\big| \leq \frac{2\delta}{\Delta - 2\delta} \;,\; \big|\frac{d^2\mu_n^*}{d\nu^2}\big| \leq \frac{2\delta}{\Delta - 2\delta}\Big(\Big(\frac{2\delta}{\Delta - 2\delta}\Big)^2 + \frac{2\delta}{\Delta - 2\delta} + 1\Big) \,.$$

□

Proposition A.2. *Under the conditions of Proposition A.1, there exist $\beta_0 \in \mathbb{R}^+$ and $p \in \mathbb{N}$ so that the following holds for all $\beta \geq \beta_0$. Let*

$$\tau'(\beta) := \tau(\beta) - \max\big\{\frac{d}{d-1}, p\big\} \,.$$

A) If $\mu + i\nu \in \mathbb{U}_0$ and $\mu \leq \mu^*(\nu;\beta)$, then

$$\big|\frac{d}{dz}\omega(\Gamma^2)(z)\big| \leq \beta C_3 \mathrm{e}^{-\tau'(\beta)|\Gamma^2|} \,.$$

B) If $\mu + i\nu \in \mathbb{U}_0$ and $\mu \geq \mu^*(\nu;\beta)$, then

$$\big|\frac{d}{dz}\omega(\Gamma^1)(z)\big| \leq \beta C_3 \mathrm{e}^{-\tau'(\beta)|\Gamma^1|} \,.$$

Proof. We consider the iterative construction of the proof of Proposition A.1 with the same choice of the sequences $\{b_n^q\}$. Suppose that $z = \mu + i\nu \in \mathbb{U}_{n-1} \backslash \mathbb{U}_n$ and $\mu \leq \mu^*(\nu;\beta)$. Suppose that $V(\Gamma^2) \leq n$. We get (see (A.29))

$$\big|\frac{d}{dz}\omega(\Gamma^2)\big| \leq \beta C_3 |\Gamma^2|^{\frac{d}{d-1}} |\omega(\Gamma^2)| \,.$$

Since by Proposition A.1 $\omega(\Gamma^2)$ is $\tau(\beta)$-stable, we get for all Γ^2 such that $V(\Gamma^2) \leq n$,

$$\big|\frac{d}{dz}\omega(\Gamma^2)\big| \leq \beta C_3 |\Gamma^2|^{\frac{d}{d-1}} \mathrm{e}^{-\tau(\beta)|\Gamma^2|} \leq \beta C_3 \mathrm{e}^{-\tau'(\beta)|\Gamma^2|} \,.$$

Suppose that $V(\Gamma^2) \geq n+1$. We estimate the derivative at z of $\omega(\Gamma^2)$ using Cauchy's formula with a circle of center z contained in $\{\mu + i\nu : \mu \leq \mu^*(\nu;\beta)\}$. We estimate from below $|\mathrm{Re} z - \mu^*(\nu;\beta)|$ when $z \in \mathbb{U}_{n-1} \backslash \mathbb{U}_n$, uniformly in ν.

$$|\mathrm{Re} z - \mu^*| \geq |\mathrm{Re} z - \mu_n^*| - |\mu_n^* - \mu^*| \geq b_n^2 - |\mu_n^* - \mu^*| \,.$$

We estimate $|\mu_n^* - \mu^*|$ by first estimating $|\mu_m^* - \mu_n^*|$. Let $m > n$; then, since $\mu_m^* \in \mathbb{U}_n$,

$$\begin{aligned} 0 &= \mathrm{Re}\big(f_2^{m-1}(\mu_m^*) - f_1^{m-1}(\mu_m^*)\big) - \mathrm{Re}\big(f_2^{n-1}(\mu_n^*) - f_1^{n-1}(\mu_n^*)\big) \\ &= \mathrm{Re}\big(f_2^{m-1}(\mu_m^*) - f_2^{n-1}(\mu_m^*)\big) - \mathrm{Re}\big(f_1^{m-1}(\mu_m^*) - f_1^{n-1}(\mu_m^*)\big) \\ &\quad + \mathrm{Re}\big(f_2^{n-1}(\mu_m^*) - f_2^{n-1}(\mu_n^*)\big) - \mathrm{Re}\big(f_1^{n-1}(\mu_m^*) - f_1^{n-1}(\mu_n^*)\big) . \end{aligned}$$

From (A.25) we get

$$|\mu_m^*(\nu;\beta) - \mu_n^*(\nu;\beta)| \le \frac{2\delta^{l(n)}}{\beta(\Delta - 2\delta)} \quad \forall\, m > n ,$$

so that

$$|\mu^*(\nu;\beta) - \mu_n^*(\nu;\beta)| \le \frac{2\delta^{l(n)}}{\beta(\Delta - 2\delta)} . \tag{A.34}$$

If $V(\Gamma^2) \ge n+1$, then $|\Gamma^2| \ge l(n+1)$. Choose $p \in \mathbb{N}$ so that for all $n \ge 1$

$$\frac{1}{|\Gamma^2|^p} \le \Big(\frac{1}{2dn^{\frac{d-1}{d}}}\Big)^p \le \frac{\chi\theta'}{(\Delta+2\delta)n^{\frac{1}{d}}} - \frac{2\delta^{l(n)}}{\beta(\Delta-2\delta)} \le b_n^2 - |\mu^* - \mu_n^*| \le |\mathrm{Re} z - \mu^*| .$$

We use Cauchy's formula with a circle of center z and radius $|\Gamma^2|^{-p}$ and get

$$\Big|\frac{d}{dz}\omega(\Gamma^2)\Big| \le |\Gamma^2|^p \mathrm{e}^{-\tau(\beta)|\Gamma^2|} \le \mathrm{e}^{-\tau'(\beta)|\Gamma^2|} .$$

□

A.3 Analytic Continuation of the Weights at μ^*

In this subsection we consider how the weight $\omega(\Gamma^2)$ for a contour with boundary condition ψ_2 behaves as function of $z = \mu + i\nu$ in the vicinity of $z^* := \mu^*(\nu;\beta) + i\nu$. We improve the domains of analyticity of the weights of contours, by making a new choice of the sequences $\{b_n^q\}$, $q = 1, 2$. The main result of this subsection is Proposition A.3. At z^* the (complex) free energies f_q, $q = 1, 2$, are well-defined and can be computed by the cluster expansion method. Moreover,

$$\mathrm{Re} f_2(z^*) = \mathrm{Re} f_1(z^*) .$$

Therefore

$$\mathrm{Re} g_1(z^*) + \mu^*(\nu;\beta) h(\psi_1) = \mathrm{Re} g_2(z^*) + \mu^*(\nu;\beta) h(\psi_2) .$$

With δ as in the proof of Proposition A.1, we get

$$|\mu^*(\nu;\beta)| \leq \frac{2\delta}{\beta\Delta},$$

and

$$|\omega(\Gamma^q)(z^*)| \leq \exp\big[-\beta\|\Gamma^q\| + \frac{2C_1\delta}{\Delta}|\Gamma^q| + \delta C_0|\Gamma^q|\big], \ \forall\, \Gamma^q.$$

We set

$$\mu^* := \mu^*(0;\beta),$$

and adopt the following convention: if a quantity, say H or f_q, is evaluated at the transition point μ^*, we simply write H^* or f_q^*.

The analyticity properties of $\omega(\Gamma^2)$ near μ^* are controlled by isoperimetric inequalities

$$V(\Gamma^2)^{\frac{d-1}{d}} \leq \chi_2(n)^{-1}\|\Gamma^2\| \quad \forall\, \Gamma^2,\, V(\Gamma^2) \geq n. \tag{A.35}$$

The difference with (A.16) is that only contours with boundary condition ψ_2 and $V(\Gamma^2) \geq n$ are considered for a given n. By definition the isoperimetric constants $\chi_2(n)$ verify

$$\chi_2(n)^{-1} := \inf\Big\{C : \frac{V(\Gamma^2)^{\frac{d-1}{d}}}{\|\Gamma^2\|} \leq C,\, \forall\, \Gamma^2 \text{ such that } V(\Gamma^2) \geq n\Big\}.$$

$\chi_2(n)$ is a bounded increasing sequence; we set $\chi_2(\infty) := \lim_n \chi_2(n)$, and define

$$R_2(n) := \inf_{m:m\leq n} \frac{\chi_2(m)}{m^{\frac{1}{d}}}.$$

There are similar definitions for $\chi_1(n)$ and $R_1(n)$. The corresponding isoperimetric inequalities control the analyticity properties of $\omega(\Gamma^1)$ around μ^*.

Lemma A.3. *For any $\chi'_q < \chi_q(\infty)$, there exists $N(\chi'_q)$ such that for all $n \geq N(\chi'_q)$,*

$$\frac{\chi'_q}{n^{\frac{1}{d}}} \leq R_q(n) \leq \frac{\chi_q(\infty)}{n^{\frac{1}{d}}}.$$

For $q = 1, 2$, $n \mapsto n^a R_q(n)$ is increasing in n, provided that $a \geq \frac{1}{d}$.

Proof. Let $q = 2$ and suppose that

$$R_2(n) = \frac{\chi_2(m)}{m^{\frac{1}{d}}} \quad \text{for } m < n.$$

Then $R_2(m') = R_2(n)$ for all $m \leq m' \leq n$. Let n' be the largest $n \geq m$ such that

$$R_2(n) = \frac{\chi_2(m)}{m^{\frac{1}{d}}}.$$

We have $n' < \infty$, otherwise

$$0 < R_2(m) = R_2(n) \leq \frac{\chi_2(\infty)}{n^{\frac{1}{d}}} \quad \forall\, n \geq m\,,$$

which is impossible. Therefore, either

$$R_2(n') = \frac{\chi_2(n')}{n'^{\frac{1}{d}}} \quad \text{or} \quad R_2(n'+1) = \frac{\chi_2(n'+1)}{(n'+1)^{\frac{1}{d}}}\,,$$

and for all $k \geq n'+1$, since $\chi_2(m)$ is increasing,

$$R_2(k) = \inf_{m\leq k} \frac{\chi_2(m)}{m^{\frac{1}{d}}} = \inf_{n'\leq m\leq k} \frac{\chi_2(m)}{m^{\frac{1}{d}}} \geq \inf_{n'\leq m\leq k} \frac{\chi_2(n')}{m^{\frac{1}{d}}} = \frac{\chi_2(n')}{k^{\frac{1}{d}}}\,. \tag{A.36}$$

Inequality (A.36) is true for infinitely many n'; since there exists m such that $\chi_2' \leq \chi_2(m)$, the first statement is proved.
On an interval of constancy of $R_2(n)$, $n \mapsto n^a R_2(n)$ is increasing. On the other hand, if on $[m_1, m_2]$

$$R_2(n) = \frac{\chi_2(n)}{n^{\frac{1}{d}}}\,,$$

then $n \mapsto n^a R_2(n)$ is increasing on $[m_1, m_2]$ since $n \mapsto \chi_2(n)$ and $n \mapsto n^{a-\frac{1}{d}}$ are increasing.

□

The next proposition gives the domains of analyticity and the stability properties of the weights $\omega(\Gamma)$ needed for estimating the derivatives of the free energy.

Proposition A.3. *Let $0 < \theta < 1$ and $0 < \varepsilon < 1$ so that $\rho(1-\theta) - \varepsilon > 0$. There exist $0 < \delta < 1$, $0 < \theta' < 1$ and $\beta_0 \in \mathbb{R}^+$, such that for all $\beta \geq \beta_0$ $\omega(\Gamma^2)$ is analytic and $\tau_1(\beta;\theta')$-stable in a complex neighborhood of*

$$\big\{z \in \mathbb{C} :\, \mathrm{Re} z \leq \mu^*(\mathrm{Im} z;\beta) + \theta\Delta^{-1} R_2(V(\Gamma^2))\big\} \cap \mathbb{U}_0\,.$$

Moreover

$$\Big|\frac{d}{dz}\omega(\Gamma^2)\Big| \leq \beta C_3 \mathrm{e}^{-\tau_2(\beta;\theta')|\Gamma^2|}\,.$$

Similar properties hold for $\omega(\Gamma^1)$ in a complex neighborhood of

$$\big\{z \in \mathbb{C} :\, \mu^*(\mathrm{Im} z;\beta) - \theta\Delta^{-1} R_1(V(\Gamma^1)) \leq \mathrm{Re} z\big\} \cap \mathbb{U}_0\,.$$

$\tau_1(\beta;\theta')$ and $\tau_2(\beta;\theta')$ are defined at (A.17) *and* (A.18).

Proof. If in the iterative method of the proof of Proposition A.1 we find $0 < \theta' < 1$ and b_n^1, b_n^2, so that (A.27), (A.28) and

$$\big(\mu^*(\nu;\beta) - \theta\Delta^{-1}R_1(n), \mu^*(\nu;\beta) + \theta\Delta^{-1}R_2(n)\big) \subset U_n(\nu;\beta) \tag{A.37}$$

hold, then Proposition A.3 is true. Formula (A.37) is satisfied if (see (A.34))

$$b_n^q \geq \theta\Delta^{-1}R_q(n) + \frac{2\delta^{l(n)}}{\beta(\Delta - 2\delta)},$$

and this is the case if

$$b_n^q := \theta\Delta^{-1}R_q(n) + \frac{C}{\beta}\delta^{n^{\frac{1}{4}}},$$

with C a suitable constant, which is chosen so that (A.21) is also satisfied. If β is large enough and δ small enough, then there exists $\theta' < 1$ so that (A.27) and (A.28) hold. Indeed, let $V(\Gamma^2) = n$, $z = \mu + i\nu$ and $\mu \leq \mu^*(\nu;\beta) + b_n^2$; then

$$\begin{aligned} -\mathrm{Re}\big(f_1^{n-1}(z) - f_2^{n-1}(z)\big)\frac{V(\Gamma^2)}{\|\Gamma^2\|} &\leq (\Delta + 2\delta)b_n^2 \frac{n^{\frac{1}{d}}}{\chi_2(n)} \\ &\leq \frac{\Delta + 2\delta}{\Delta}\theta + \frac{C}{\beta}\delta^{n^{\frac{1}{4}}}\frac{n^{\frac{1}{d}}}{\chi_2(n)} \leq \theta'. \end{aligned}$$

□

A.4 Derivatives of the Free Energy

Although non-analytic behavior of the free energy occurs only in the thermodynamical limit, most of the analysis is done at finite volume. We write

$$[g]_{t'}^{(k)} := \frac{\mathrm{d}^k}{\mathrm{d}t^k}g(t)\bigg|_{t=t'}$$

for the k^{th} order derivative at t' of the function g. The method of Isakov [Isakov1] allows to get estimates of the derivatives of the free energy at μ^*, which are *uniform in the volume*. We consider the case of the boundary condition ψ_2. The other case is similar. We tacitly assume that β is large enough so that Lemma A.1 and all results of subsections A.2 and A.3 are valid. The main tool for estimating the derivatives of the free energy is Cauchy's formula. However, we need to establish several results before we can obtain the desired estimates on the derivatives of the free energy. The preparatory work is done in this subsection, which is divided into three subsections. In A.4 we give an expression of the derivatives of the free energy in terms of the derivatives of a free energy of a contour $u(\Gamma^2) = -\log(1 + \phi_\Lambda(\Gamma^2)) \approx -\phi_\Lambda(\Gamma^2)$ (see (A.39)). The main work is to estimate

$$\frac{k!}{2\pi i}\oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\,dz\,.$$

The boundary of the disc D_r is decomposed naturally into two parts, ∂D_r^g and ∂D_r^d, and the integral into two integrals $I^g_{k,n}(\Gamma^2)$ and $I^d_{k,n}(\Gamma^2)$ (see (A.41) and (A.42)). In A.4 we prove the upper bound (A.43) for $I^g_{k,n}(\Gamma^2)$, and in A.4 we evaluate $I^d_{k,n}(\Gamma^2)$ by the stationary phase method, see (A.47) and (A.48). This is a key point in the proof of Theorem A.1, since we obtain lower and upper bounds for $I^d_{k,n}(\Gamma^2)$.

An expression for the derivatives of the free energy

Let $\Lambda = \Lambda(L)$ be the cubic box

$$\Lambda(L) := \{x \in \mathbf{Z}^d : |x| \leq L\}\,.$$

We introduce a linear order, denoted by $\leq$, among all contours $\Gamma^q \subset \Lambda$ with boundary condition ψ_q. We assume that the linear order is such that $V(\Gamma'^q) \leq V(\Gamma^q)$ if $\Gamma'^q \leq \Gamma^q$. There exists a natural enumeration of the contours by the positive integers. The predecessor of Γ^q in that enumeration (if Γ^q is not the smallest contour) is denoted by $i(\Gamma^q)$. We introduce the restricted partition function $\Theta_{\Gamma^q}(\Lambda)$, which is computed with the contours of

$$\mathcal{C}_\Lambda(\Gamma^q) := \{\Gamma'^q \subset \Lambda : \Gamma'^q \leq \Gamma^q\}\,,$$

that is

$$\Theta_{\Gamma^q}(\Lambda) := 1 + \sum \prod_{i=1}^n \omega(\Gamma^q_i)\,, \tag{A.38}$$

where the sum is over all families of compatible contours $\{\Gamma^q_1, \ldots, \Gamma^q_n\}$ which belong to $\mathcal{C}_\Lambda(\Gamma^q)$. The partition function $\Theta_q(\Lambda)$ is written as a finite product

$$\Theta_q(\Lambda) = \prod_{\Gamma^q \subset \Lambda} \frac{\Theta_{\Gamma^q}(\Lambda)}{\Theta_{i(\Gamma^q)}(\Lambda)}\,.$$

By convention $\Theta_{i(\Gamma^q)}(\Lambda) := 1$ when Γ^q is the smallest contour. We set

$$u_\Lambda(\Gamma^q) := -\log \frac{\Theta_{\Gamma^q}(\Lambda)}{\Theta_{i(\Gamma^q)}(\Lambda)}\,.$$

$u_\Lambda(\Gamma^q)$ is the free energy cost for introducing the new contour Γ^q in the restricted model, where all contours verify $\Gamma'^q \leq \Gamma^q$. We have the identity

$$\begin{aligned}\Theta_{\Gamma^q}(\Lambda) &= \Theta_{i(\Gamma^q)}(\Lambda) + \omega(\Gamma^q)\,\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q)) \\ &= \Theta_{i(\Gamma^q)}(\Lambda)\left(1 + \omega(\Gamma^q)\,\frac{\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))}{\Theta_{i(\Gamma^q)}(\Lambda)}\right)\,.\end{aligned}$$

In this last expression $\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))$ denotes the restricted partition function

$$\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q)) := 1 + \sum \prod_{i=1}^{n} \omega(\Gamma_i^q) \,,$$

where the sum is over all families of compatible contours $\{\Gamma_1^q, \ldots, \Gamma_n^q\}$ which belong to $\mathcal{C}_\Lambda(i(\Gamma^q))$, and such that $\{\Gamma^q, \Gamma_1^q, \ldots, \Gamma_n^q\}$ is a compatible family. We also set

$$\phi_\Lambda(\Gamma^q) := \omega(\Gamma^q) \, \frac{\Theta_{i(\Gamma^q)}(\Lambda(\Gamma^q))}{\Theta_{i(\Gamma^q)}(\Lambda)} \,.$$

With these notations

$$u_\Lambda(\Gamma^q) = -\log\big(1 + \phi_\Lambda(\Gamma^q)\big) = \sum_{n \geq 1} \frac{(-1)^n}{n} \phi_\Lambda(\Gamma^q)^n \,, \tag{A.39}$$

and for $k \geq 2$

$$|\Lambda| \beta [f_\Lambda^q]_{\mu^*}^{(k)} = \sum_{\Gamma^q \subset \Lambda} [u_\Lambda(\Gamma^q)]_{\mu^*}^{(k)} .$$

We consider the case of the boundary condition ψ_2. $[\phi_\Lambda(\Gamma^2)^n]_{\mu^*}^{(k)}$ is computed using Cauchy's formula,

$$[\phi_\Lambda(\Gamma^2)^n]_{\mu^*}^{(k)} = \frac{k!}{2\pi i} \oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z - \mu^*)^{k+1}} \, dz \,,$$

where ∂D_r is the boundary of a disc D_r of radius r and center μ^* inside the analyticity region of Proposition A.3,

$$\mathbb{U}_0 \cap \big\{ z \in \mathbb{C} : \ \mathrm{Re} z \leq \mu^*(\mathrm{Im}(z); \beta) + \theta \Delta^{-1} R_2(V(\Gamma^2)) \big\} \,.$$

The function $z \mapsto \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}$ is real on the real axis, so that

$$\overline{\frac{\phi_\Lambda(\Gamma^2)^n(\overline{z})}{(\overline{z} - \mu^*)^{k+1}}} = \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z - \mu^*)^{k+1}} \,,$$

and consequently

$$\frac{k!}{2\pi i} \oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z - \mu^*)^{k+1}} \, dz = \mathrm{Re} \Big\{ \frac{k!}{2\pi i} \oint_{\partial D_r} \frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z - \mu^*)^{k+1}} \, dz \Big\} \,. \tag{A.40}$$

Remark A.2. From Lemma A.2, there exists C' independent of ν and n, so that

$$\mu_n^*(\nu;\beta) \geq \mu_n^*(0;\beta) - C'\nu^2 \,.$$

This implies that the region $\{\mathrm{Re} z \leq \mu^* - C'(\mathrm{Im} z)^2 + \theta\Delta^{-1}R_2(V(\Gamma^2))\}$ is always in the analyticity region of $\omega(\Gamma^2)$, which is given in Proposition A.3. Therefore, if

$$C' \leq \frac{1}{2\big(\theta\Delta^{-1}R_2(V(\Gamma^2))\big)^2}\,,$$

then the disc D_r of center μ^* and radius $r = \theta\Delta^{-1}R_2(V(\Gamma^2))$ is inside the analyticity region of $\omega(\Gamma^2)$. This happens as soon as $V(\Gamma^2)$ is large enough.

Assuming that the disc D_r is inside the analyticity region of $\omega(\Gamma^2)$, we decompose ∂D_r into

$$\partial D_r^g := \partial D_r \cap \{z:\ \mathrm{Re} z \leq \mu^*(\mathrm{Im}(z);\beta) - \theta\Delta^{-1}R_1(V(\Gamma^2))\}\,,$$

and

$$\partial D_r^d := \partial D_r \cap \{z:\ \mathrm{Re} z \geq \mu^*(\mathrm{Im}(z);\beta) - \theta\Delta^{-1}R_1(V(\Gamma^2))\}\,,$$

and write (A.40) as a sum of two integrals $I_{k,n}^g(\Gamma^2)$ and $I_{k,n}^d(\Gamma^2)$ (see figure A.1),

$$I_{k,n}^g(\Gamma^2) := \mathrm{Re}\Big\{\frac{k!}{2\pi i}\oint_{\partial D_r^g}\frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\,dz\Big\} \tag{A.41}$$

and

$$I_{k,n}^d(\Gamma^2) := \mathrm{Re}\Big\{\frac{k!}{2\pi i}\oint_{\partial D_r^d}\frac{\phi_\Lambda(\Gamma^2)^n(z)}{(z-\mu^*)^{k+1}}\,dz\Big\}\,. \tag{A.42}$$

An upper bound for $I_{k,n}^g(\Gamma^2)$

$I_{k,n}^g(\Gamma^2)$ is not the main contribution to (A.40), so that it is sufficient to get an upper bound for this integral. Let $z \in \mathbb{U}_0$ and $\mathrm{Re} z \leq \mu^*\big(\mathrm{Im}(z);\beta\big)$. We set

$$\overline{\Gamma^2} := \{x \in \mathbf{Z}^d:\ d(x, \mathrm{supp}\,\Gamma^2) \leq 1\}\,.$$

There exists a constant C_5 such that $|\overline{\Gamma^2}| \leq C_5|\Gamma^2|$. From (A.31) we get

$$|\omega(\Gamma^2)| \leq \exp\big[-\beta\|\Gamma^2\| + \beta|\mathrm{Re} z|C_1|\Gamma^2| + 3C_0\delta|\Gamma^2|\big]\,,$$

and by the cluster expansion method

$$\Big|\frac{\Theta_{i(\Gamma^2)}(\Lambda(\Gamma^2))}{\Theta_{i(\Gamma^2)}(\Lambda)}\Big| \leq \mathrm{e}^{\delta|\overline{\Gamma^2}|} \leq \mathrm{e}^{\delta C_5|\Gamma^2|}\,.$$

We set

$$\zeta := z - \mu^* .$$

Therefore, there exists a constant C_6 so that

$$|\phi_\Lambda(\Gamma^2)| \le \mathrm{e}^{-\beta\|\Gamma^2\|(1-C_6\delta-|\mathrm{Re}\zeta|C_1\rho^{-1})} \quad \text{if} \quad \mathrm{Re}\zeta \le \mu^*\big(\mathrm{Im}(\zeta);\beta\big) - \mu^* .$$

This upper bound implies

$$I^g_{k,n}(\Gamma^2) \le \frac{k!}{r^k}\,\mathrm{e}^{-n\beta\|\Gamma^2\|(1-C_6\delta-rC_1\rho^{-1})} . \tag{A.43}$$

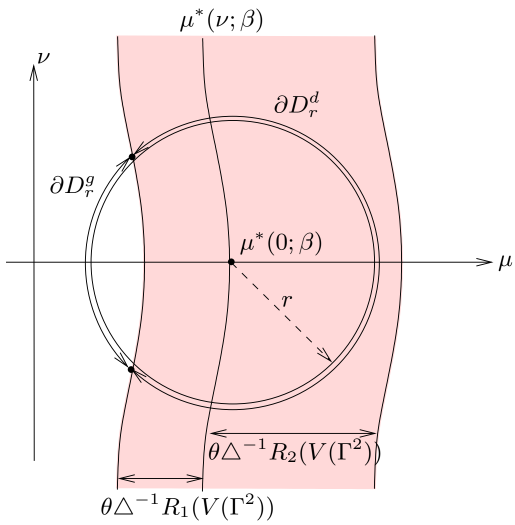

Figure A.1: The decomposition of the integral into $I^g_{k,n}(\Gamma^2)$ and $I^d_{k,n}(\Gamma^2)$

Lower and upper bounds for $I^d_{k,n}(\Gamma^2)$

In order to apply the stationary phase method to evaluate $I^d_{k,n}(\Gamma^2)$, we first rewrite $\phi_\Lambda(\Gamma^2)$ in the following form,

$$\phi_\Lambda(\Gamma^2)(z) = \phi^*_\Lambda(\Gamma^2)\,\mathrm{e}^{\beta\Delta V(\Gamma^2)(\zeta+\mathsf{g}(\Gamma^2)(\zeta))} ,$$

where $\mathsf{g}(\Gamma^2)$ is an analytic function of ζ in a neighborhood of $\zeta = 0$ and $\mathsf{g}(\Gamma^2)(0) = 0$. Let

$$\mu^*\big(\mathrm{Im}(z);\beta\big) - \theta\Delta^{-1}R_1(V(\Gamma^2)) \le \mathrm{Re}z \le \mu^*\big(\mathrm{Im}(z);\beta\big) + \theta\Delta^{-1}R_2(V(\Gamma^2)) .$$

In this region (see figure A.1) we control the weights of contours with boundary conditions ψ_2 and ψ_1, whose volume is smaller than $V(\Gamma^2)$. By the cluster expansion method there exists an analytic function $\mathsf{g}(\Gamma^2)$, which is real on the real axis, so that

$$\begin{aligned}\phi_\Lambda(\Gamma^2) &= \exp\Big[-\beta H(\varphi_{\Gamma^2}|\psi_2) + \underbrace{\log\frac{\Theta_1(\mathrm{Int}_1\,\Gamma^2)}{\Theta_2(\mathrm{Int}_1\,\Gamma^2)} + \log\frac{\Theta_{i(\Gamma^2)}(\Lambda(\Gamma^2))}{\Theta_{i(\Gamma^2)}(\Lambda)}}_{:=\mathtt{G}(\Gamma^2)(z)}\Big] \\ &= \phi^*_\Lambda(\Gamma^2)\exp\Big[\beta\Delta V(\Gamma^2)\zeta + \underbrace{\int_{\mu^*}^{\mu^*+\zeta}\Big(\frac{d}{dz}\mathtt{G}(\Gamma^2)(z) - \beta a(\varphi_{\Gamma^2})\Big)dz}_{:=\beta\Delta V(\Gamma^2)\mathsf{g}(\Gamma^2)(\zeta)}\Big].\end{aligned}$$

For large enough β, $\tau'(\beta) \geq \tau_2(\beta;\theta')$, so that we get from Lemma A.1 and Propositions A.1 to A.3

$$\begin{aligned}\Big|\frac{d}{d\zeta}\mathsf{g}(\Gamma^2)(\zeta)\Big| &\leq 2C_3K\mathrm{e}^{-\tau_2(\beta;\theta')}\Big(\frac{1}{\Delta} + \frac{C_0|\Gamma^2|}{\Delta V(\Gamma^2)} + \frac{|\overline{\Gamma^2}|}{\Delta V(\Gamma^2)}\Big) + \frac{C_1|\Gamma^2|}{\Delta V(\Gamma^2)} \\ &\leq C_7\,\mathrm{e}^{-\tau_2(\beta;\theta')} + C_8\frac{|\Gamma^2|}{V(\Gamma^2)}\,, \qquad \text{(A.44)}\end{aligned}$$

for suitable constants C_7 and C_8. Moreover, there exists a constant C_9 so that

$$\exp\big[-\beta\|\Gamma^2\|(1+C_9\delta)\big] \leq \phi^*_\Lambda(\Gamma^2) \leq \exp[-\beta\|\Gamma^2\|(1-C_9\delta)]\,. \qquad \text{(A.45)}$$

Let

$$c(n) := n\beta\Delta V(\Gamma^2)\,.$$

We parametrize ∂D^d_r by $z := \mu^* + r\mathrm{e}^{i\alpha}$, $-\alpha_1 \leq \alpha \leq \alpha_2$, $0 < \alpha_i \leq \pi$.

$$I^d_{k,n}(\Gamma^2) = k!\frac{\phi^*_\Lambda(\Gamma^2)^n}{2\pi r^k}\int_{-\alpha_1}^{\alpha_2}\mathrm{e}^{c(n)r\cos\alpha + c(n)\mathrm{Re}\,\mathsf{g}(\Gamma^2)(\zeta)}\big[\cos(\widetilde{\psi}(\alpha))\big]\,d\alpha\,,$$

where

$$\widetilde{\psi}(\alpha) := c(n)r\sin\alpha + c(n)\,\mathrm{Im}\,\mathsf{g}(\Gamma^2)(\zeta) - k\alpha\,.$$

We search for a stationary phase point $\zeta_{k,n} = r_{k,n}\mathrm{e}^{i\alpha_{k,n}}$ defined by the equations

$$\frac{d}{d\alpha}\Big(c(n)r\cos\alpha + c(n)\mathrm{Re}\,\mathsf{g}(\Gamma^2)\big(r\mathrm{e}^{i\alpha}\big)\Big) = 0 \quad \text{and} \quad \frac{d}{d\alpha}\widetilde{\psi}(\alpha) = 0\,.$$

These equations are equivalent to the equations ($'$ denotes the derivative with respect to ζ)

$$\begin{aligned}c(n)\sin\alpha\big(1 + \mathrm{Re}\,\mathsf{g}(\Gamma^2)'(\zeta)\big) + \cos\alpha\mathrm{Im}\,\mathsf{g}(\Gamma^2)'(\zeta) &= 0\,; \\ c(n)r\cos\alpha\big(1 + \mathrm{Re}\,\mathsf{g}(\Gamma^2)'(\zeta)\big) - r\sin\alpha\mathrm{Im}\,\mathsf{g}(\Gamma^2)'(\zeta) &= k\,.\end{aligned}$$

Since $\mathsf{g}(\Gamma^2)$ is real on the real axis, $\alpha_{k,n} = 0$ and $r_{k,n}$ is solution of

$$c(n)r\big(1 + \mathsf{g}(\Gamma^2)'(r)\big) = k\,. \qquad \text{(A.46)}$$

Lemma A.4. *Let $\alpha_i \geq \pi/4$, $i = 1, 2$, $A \leq 1/25$ and $c(n) \geq 1$. If $\mathsf{g}(\zeta)$ is analytic in ζ in the disc $\{\zeta : |\zeta| \leq R\}$, real on the real axis, and for all ζ in that disc*

$$\Big|\frac{d}{d\zeta}\mathsf{g}(\Gamma^2)(\zeta)\Big| \leq A\,,$$

then there exists $k_0(A) \in \mathbb{N}$, such that for all integers k,

$$k \in \big[k_0(A), c(n)(1-2\sqrt{A})R\big]\,,$$

there is a unique solution $0 < r_{k,n} < R$ of (A.46)*. Moreover,*

$$\begin{aligned}\frac{\mathrm{e}^{cr_{k,n}+c(n)\,\mathsf{g}(\Gamma^2)(r_{k,n})}}{10\sqrt{c(n)r_{k,n}}} &\leq \frac{1}{2\pi}\int_{-\alpha_1}^{\alpha_2} \mathrm{e}^{c(n)r\cos\alpha+c(n)\mathrm{Re}\,\mathsf{g}(\Gamma^2)}\left[\cos(\widetilde{\psi}(\alpha))\right]d\alpha \\ &\leq \frac{\mathrm{e}^{c(n)r_{k,n}+c(n)\,\mathsf{g}(\Gamma^2)(r_{k,n})}}{\sqrt{c(n)r_{k,n}}}\,.\end{aligned}$$

Proof. Existence and uniqueness of $r_{k,n}$ is a consequence of the monotonicity of $r \mapsto c(n)r\big(1+\mathsf{g}(\Gamma^2)'(r)\big)$. The last part of Lemma A.4 is proven in appendix of [Isakov1]. The computation is relatively long, but standard [3].

□

Setting $c(n) = n\beta\Delta V(\Gamma^2)$ and $R = \theta\Delta^{-1}R_2(V(\Gamma^2))$ in Lemma A.4 we get sufficient conditions for the existence of a stationary phase point and the following evaluation of the integral $I^d_{k,n}(\Gamma^2)$ by that method. Since $r_{k,n}$ is solution of (A.46), we have

$$k - \frac{kA}{(1+A)} = \frac{k}{(1+A)} \leq c(n)r_{k,n} \leq \frac{k}{(1-A)} = k + \frac{kA}{(1-A)}\,,$$

and

$$c(n)|\mathsf{g}(\Gamma^2)(r_{k,n})| = c(n)\left|\int_0^{r_{k,n}} \mathsf{g}(\Gamma^2)'(\zeta)d\zeta\right| \leq Ac(n)r_{k,n} \leq k\frac{A}{1-A}\,.$$

Therefore Lemma A.4 implies

$$\begin{aligned}\frac{\sqrt{1-A}}{10\sqrt{k}}c_-^k\, c(n)^k\,\frac{k!\,\mathrm{e}^k}{k^k}\,\phi^*_\Lambda(\Gamma^2)^n &\leq I^d_{k,n}(\Gamma^2) \\ &\leq \frac{\sqrt{1+A}}{\sqrt{k}}c_+^k\, c(n)^k\,\frac{k!\,\mathrm{e}^k}{k^k}\,\phi^*_\Lambda(\Gamma^2)^n\,,\end{aligned} \tag{A.47}$$

with

$$c_+(A) := (1+A)\exp\left[\frac{2A}{1-A}\right], \quad c_-(A) := (1-A)\exp\left[-\frac{2A}{1-A^2}\right], \tag{A.48}$$

[3] The details can be found in Appendix B.

If A converges to 0, then $c_{\pm}$ converges to 1. We assume that (see (A.44))

$$C_7\,\mathrm{e}^{-\tau_2(\beta;\theta')} \leq \frac{A}{2} \quad \text{and} \quad C_8\frac{|\Gamma^2|}{V(\Gamma^2)} \leq \frac{A}{2}\,. \tag{A.49}$$

A can be chosen as small as we wish, provided that β is large enough and $\frac{|\Gamma^2|}{V(\Gamma^2)}$ small enough.

A.4.1 Bounds at Finite Volume

We estimate the derivative of $[f^2_\Lambda]^{(k)}_{\mu^*}$ for large enough k. The main result of this subsection is Proposition A.4.
Let $0<\theta<1$, $A\leq 1/25$, and set

$$\hat{\theta} := \theta(1-2\sqrt{A})\,.$$

Let $\varepsilon'>0$ and χ_2' so that

$$(1+\varepsilon')\chi_2' > \chi_2(\infty)\,. \tag{A.50}$$

The whole analysis depends on the parameters θ and ε'. We fix the values of θ, and ε' by the following conditions, which are needed for the proof of Proposition A.4. We choose $0<A_0<1/25$, θ and ε' so that

$$\mathrm{e}^{\frac{1}{d}}\,\frac{1}{\theta(1-2\sqrt{A_0})} < \frac{d}{d-1}\,\frac{c_-(A_0)^{\frac{d-1}{d}}}{1+\varepsilon'} \quad \text{and} \quad \frac{1-2\sqrt{A_0}}{1+\varepsilon'}\,\frac{d}{d-1} > 1\,. \tag{A.51}$$

This is possible, since

$$\frac{d}{(d-1)\,\mathrm{e}^{\frac{1}{d}}} > 1\,.$$

Indeed,

$$\begin{aligned}
d\Big(\mathrm{e}^{\frac{1}{d}}-1\Big) &= d\Big(\mathrm{e}^{\frac{1}{d}}-1-\frac{1}{d}+\frac{1}{d}\Big) = \sum_{n\geq 2}\frac{1}{n!}\Big(\frac{1}{d}\Big)^{n-1}+1 \\
&= 1+\sum_{n\geq 1}\frac{1}{(n+1)!}\Big(\frac{1}{d}\Big)^{n} \\
&< 1-\frac{1}{2d}+\sum_{n\geq 1}\frac{1}{n!}\Big(\frac{1}{d}\Big)^{n} = \mathrm{e}^{\frac{1}{d}}-\frac{1}{2d}\,.
\end{aligned}$$

Notice that conditions (A.51) are still verified with the same values of θ and ε' if we replace A_0 by $0<A<A_0$. Given θ, the value of θ' is fixed in Proposition A.3. From now we assume that β is so large that all results of subsections A.2 and A.3 are valid. The value of $0<A<A_0$ is fixed in the proof of Lemma A.6.

Given k large enough, there is a natural distinction between contours Γ^2 such that $\hat{\theta}\beta V(\Gamma^2)R_2(V(\Gamma^2)) \leq k$ and those such that $\hat{\theta}\beta V(\Gamma^2)R_2(V(\Gamma^2)) > k$. For the latter we can estimate $I^d_{k,n}(\Gamma^2)$ by the stationary phase method. We need as a matter of fact a finer distinction between contours. We distinguish three classes of contours:

1. k-small contours: $\hat{\theta}\beta V(\Gamma^2)R_2(V(\Gamma^2)) \leq k$;
2. fat contours: for $\eta \geq 0$, fixed later by (A.54), $V(\Gamma^2)^{\frac{d-1}{d}} \leq \eta \,\|\Gamma^2\|$;
3. k-large and thin contours: $\hat{\theta}\beta V(\Gamma^2)R_2(V(\Gamma^2)) > k$, $V(\Gamma^2)^{\frac{d-1}{d}} > \eta\,\|\Gamma^2\|$.

We make precise the meaning of k *large enough*. By Lemma A.3 $V \mapsto VR_2(V)$ is increasing in V, and there exists $N(\chi_2')$ such that

$$R_2(V) \geq \frac{\chi_2'}{V^{\frac{1}{d}}} \quad \text{if} \quad V \geq N(\chi_2')\,.$$

We assume that there is a k-small contour Γ^2 such that $V(\Gamma^2) \geq N(\chi_2')$, and that the maximal volume of the k-small contours is so large that remark A.2 is valid. We also assume (see Lemma A.4) that $k > k_0(A)$ and that for a k-large and thin contour (see (A.44) and (A.49))

$$C_8 \frac{|\Gamma^2|}{V(\Gamma^2)} \leq \frac{C_8}{\rho\eta V(\Gamma^2)^{\frac{1}{d}}} \leq \frac{A}{2}\,,$$

so that $|\mathsf{g}(\Gamma^2)'| \leq A$, and

$$\frac{C_1 k}{\rho\Delta(1-A_0)\eta V(\Gamma^2)^{\frac{1}{d}}} \leq \frac{k}{10} \tag{A.52}$$

are verified. There exists $K(A,\eta,\beta)$ such that if $k \geq K(A,\eta,\beta)$, then k is *large enough*. From now on $k \geq K(A,\eta,\beta)$.

Contribution to $[f^q_\Lambda]^{(k)}_{\mu^*}$ from the k-small and fat contours

Let Γ^2 be a k-small contour. Since $V \mapsto R_2(V)$ is decreasing in V, $u_\Lambda(\Gamma^2)$ is analytic in the region

$$\{z:\, \mathrm{Re} z \leq \mu^*(\mathrm{Im} z;\beta) + \theta\Delta^{-1}R_2(V^*)\} \cap \mathbb{U}_0\,,$$

where V^* is the maximal volume of k-small contours. V^* satisfies

$$V^{*\frac{d-1}{d}} \leq \frac{k}{\hat{\theta}\beta\chi_2'}\,.$$

Hence

$$\theta\Delta^{-1}R_2(V^*) \geq \hat{\theta}\Delta^{-1}\chi_2' V^{*-\frac{1}{d}} \geq \Delta^{-1}\big(\hat{\theta}\chi_2'\big)^{\frac{d}{d-1}}\beta^{\frac{1}{d-1}}k^{-\frac{1}{d-1}}\,.$$

Since remark A.2 is valid, we estimate the derivative of $u_\Lambda(\Gamma^2)$ by Cauchy's formula with a disc centered at μ^* with radius $\Delta^{-1}\big(\hat{\theta}\chi_2'\big)^{\frac{d}{d-1}}\beta^{\frac{1}{d-1}}k^{-\frac{1}{d-1}}$. There exists a constant C_{10} such that

$$\Big|\sum_{\substack{\Gamma^2:\mathrm{Int}\,\Gamma^2\ni 0\\ V(\Gamma^2)^{\frac{d-1}{d}}\leq\frac{k}{\hat{\theta}\beta\chi_2'}}}[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}\Big| \leq C_{10}\Big(\frac{\Delta}{\beta^{\frac{1}{d-1}}(\hat{\theta}\chi_2')^{\frac{d}{d-1}}}\Big)^k k!\,k^{\frac{k}{d-1}}\,. \tag{A.53}$$

Let Γ^2 be a fat contour, which is not k-small. We use in Cauchy's formula a disc centered at μ^* with radius

$$\hat{\theta}\Delta^{-1}\chi_2(1)V(\Gamma^2)^{-\frac{1}{d}} \leq \theta\Delta^{-1}R_2(V(\Gamma^2))\,.$$

We get (see (A.10))

$$\begin{aligned}\big|[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}\big| &\leq k!\,\left(\frac{\Delta V(\Gamma^2)^{\frac{1}{d}}}{\chi_2(1)\hat{\theta}}\right)^k \mathrm{e}^{-n[\tau_1(\beta;\theta')-C_5\delta]|\Gamma^2|}\\ &\leq k!\,\left(\frac{\Delta\,(C_2\eta)^{\frac{1}{d-1}}}{\chi_2(1)\hat{\theta}}\right)^k |\Gamma^2|^{\frac{k}{d-1}}\mathrm{e}^{-n[\tau_1(\beta;\theta')-C_5\delta]|\Gamma^2|}\,.\end{aligned}$$

We sum over n and over Γ^2 using the inequality

$$\sum_{m\geq 1} m^p\,\mathrm{e}^{-qm} \leq \frac{1}{q^p}\,\Gamma(p+1) \quad (p\geq 2\,,\ q\geq 2)\,.$$

There exist C_{11} and $C_{12}(\theta')>0$ so that

$$\begin{aligned}\sum_{\substack{\Gamma^2:\mathrm{Int}\,\Gamma^d\ni 0\\ V(\Gamma^2)^{\frac{d-1}{d}}\leq\eta\|\Gamma^2\|\\ \Gamma^2\text{ not }k\text{-small}}}\big|[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}\big| &\leq C_{11}\left(\frac{\Delta\,(C_2\eta)^{\frac{1}{d-1}}}{(C_{12}\beta)^{\frac{1}{d-1}}\chi_2(1)\hat{\theta}}\right)^k k!\,\Gamma\Big(\frac{k}{d-1}+1\Big)\\ &\leq C_{11}\left(\frac{\Delta\,(C_2\eta)^{\frac{1}{d-1}}}{(C_{12}\beta)^{\frac{1}{d-1}}\chi_2(1)\hat{\theta}}\right)^k k!\;k^{\frac{k}{d-1}}\,.\end{aligned}$$

We choose η so small that (see (A.53))

$$\frac{\Delta\,(C_2\eta)^{\frac{1}{d-1}}}{(C_{12}\beta)^{\frac{1}{d-1}}\chi_2(1)\hat{\theta}} < \frac{\Delta}{\beta^{\frac{1}{d-1}}(\hat{\theta}\chi_2(\infty))^{\frac{d}{d-1}}} < \frac{\Delta}{\beta^{\frac{1}{d-1}}(\hat{\theta}\chi_2')^{\frac{d}{d-1}}}\,. \tag{A.54}$$

Contribution to $[f^q_\Lambda]^{(k)}_{\mu^*}$ from the k-large and thin contours

For k-large and thin contours we get lower and upper bounds for $[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}$. There are two cases.

A. Assume that $R_1(V(\Gamma^2)) \geq R_2(V(\Gamma^2))$, or that $V(\Gamma^2)$ is so large that

$$\hat{\theta}\beta V(\Gamma^2) R_1(V(\Gamma^2)) > k\,.$$

For each $n \geq 1$ let $c(n) = n\beta\Delta V(\Gamma^2)$. Under these conditions we can apply Lemma A.4 with a disc $D_{r_{k,n}}$ so that $\partial D_{r_{k,n}} = \partial D^d_{r_{k,n}}$. Indeed, if $R_1(V(\Gamma^2)) \geq R_2(V(\Gamma^2))$, then we apply Lemma A.4 with $R = \theta\Delta^{-1}R_2(V(\Gamma^2))$, and in the other case we set $R = \theta\Delta^{-1}R_1(V(\Gamma^2))$. In both cases $r_{k,n} < R$, which implies $\partial D_{r_{k,n}} = \partial D^d_{r_{k,n}}$. Therefore we get for $I^d_{k,n}(\Gamma^2)$ the lower and upper bounds (A.47).

Lemma A.5. *There exists a function $D(k)$, $\lim_{k\to\infty} D(k) = 0$, such that for β sufficiently large and A sufficiently small the following holds. If $k \geq K(A,\eta,\beta)$ and $R_1(V(\Gamma^2)) \geq R_2(V(\Gamma^2))$ or $\hat{\theta}\beta V(\Gamma^2)R_1(V(\Gamma^2)) > k$, then*

$$(1 - D(k))\,[\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \leq -[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \leq (1 + D(k))\,[\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}\,.$$

Proof. We have

$$-[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} = [\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} + [\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \sum_{n\geq 2} \frac{(-1)^{(n-1)}}{n} \frac{[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}}{[\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}}\,.$$

From (A.47) there exists a constant C_{13},

$$\frac{[\phi_\Lambda(\Gamma^2)^n]^{(k)}_{\mu^*}}{[\phi_\Lambda(\Gamma^2)]^{(k)}_{\mu^*}} \leq C_{13}\,\phi^*_\Lambda(\Gamma^2)^{(n-1)} \left(\frac{c_+}{c_-}\right)^k n^k\,.$$

The isoperimetric inequality (A.35), $R_2(n) \leq \chi_2(n) n^{-\frac{1}{d}}$ and the definition of k-large volume contour imply

$$\beta\|\Gamma^2\| \geq \beta\chi_2(V(\Gamma^2))V(\Gamma^2)^{\frac{d-1}{d}} \geq \hat{\theta}\beta R_2(V(\Gamma^2))V(\Gamma^2) \geq k\,.$$

Let $b := C_9\delta$ (see (A.45)); we may assume $\frac{9}{10} - b \geq \frac{4}{5}$ by taking β large enough. Then

$$\begin{aligned}
\frac{c_+^k}{c_-^k}\sum_{n\geq 2} n^{k-1}\mathrm{e}^{-(n-1)(1-b)k} &\leq \frac{c_+^k}{c_-^k}\sum_{n\geq 2} \mathrm{e}^{-\frac{1}{10}(n-1)k}\mathrm{e}^{-k\left[(\frac{9}{10}-b)(n-1)-\ln n\right]} \\
&\leq \frac{c_+^k}{c_-^k}\sum_{n\geq 2} \mathrm{e}^{-\frac{1}{10}(n-1)k}\mathrm{e}^{-k\left[\frac{4}{5}(n-1)-\ln n\right]} \\
&\leq \left(\frac{c_+}{c_-}\mathrm{e}^{-\frac{1}{10}}\right)^k \sum_{n\geq 1} \mathrm{e}^{-\frac{1}{10}nk}\,.
\end{aligned}$$

We choose A so small that $c_+(A)c_-(A)^{-1}\,\mathrm{e}^{-\frac{1}{10}} \leq 1$.

□

B. The second case is when

$$\hat{\theta}\beta V(\Gamma^2)R_1(V(\Gamma^2)) \leq k \leq \hat{\theta}\beta V(\Gamma^2)R_2(V(\Gamma^2))\,.$$

Since the contours are also thin,

$$\begin{aligned}\beta\|\Gamma^2\| &\leq \eta^{-1}\hat{\theta}^{-1}\chi_1(1)^{-1}\beta\hat{\theta}\chi_1(1)V(\Gamma^2)^{\frac{d-1}{d}}\\ &\leq \eta^{-1}\hat{\theta}^{-1}\chi_1(1)^{-1}\beta\hat{\theta}V(\Gamma^2)R_1(V(\Gamma^2))\\ &\leq \eta^{-1}\hat{\theta}^{-1}\chi_1(1)^{-1}k \equiv \lambda k\,.\end{aligned}$$

We choose $R = \beta\Delta^{-1}R_2(V(\Gamma^2))$ in Lemma A.4. The integration in (A.40) is decomposed into two parts (see figure A.1). We show that the contribution from the integration over $\partial D^g_{r_{k,n}}$ is negligible for large enough β. Since $k \geq K(A,\eta,\beta)$ and the contours verify $V(\Gamma^2)^{\frac{d-1}{d}} > \eta\|\Gamma^2\|$, we have

$$n\beta\|\Gamma^2\|r_{k,n} \leq \frac{k}{\Delta(1-A)\eta V(\Gamma^2)^{\frac{1}{d}}} \leq \frac{k}{\Delta(1-A_0)\eta V(\Gamma^2)^{\frac{1}{d}}}\,.$$

By definition of $K(A,\eta,\beta)$ (see (A.52))

$$n\beta\|\Gamma^2\|\rho^{-1}C_1 r_{k,n} \leq \frac{k}{10}\,.$$

From (A.43) with $r = r_{k,n}$ we obtain that the contribution to $|[u_\Lambda(\Gamma^q)]^{(k)}_{\mu^*}|$ is at most

$$(1+A)^k\big(\beta\Delta V(\Gamma^2)\big)^k \exp\big(\frac{k}{10}\big)\frac{k!}{k^k}\sum_{n\geq 1} n^k \mathrm{e}^{-n\beta\|\Gamma^2\|(1-C_6\delta)}\,.$$

As in the proof of Lemma A.5, we choose β large enough so that we can assume that $\frac{9}{10} - C_6\delta \geq \frac{4}{5}$. Then

$$\begin{aligned}\sum_{n\geq 1} n^k \mathrm{e}^{-n\beta\|\Gamma^2\|(1-C_6\delta)} &\leq \mathrm{e}^{-\beta\|\Gamma^2\|(1-C_6\delta)}\Big(1+\sum_{n\geq 2}\mathrm{e}^{-\frac{1}{10}(n-1)k}\mathrm{e}^{-k\left[\frac{4}{5}(n-1)-\ln n\right]}\Big)\\ &\leq \mathrm{e}^{-\beta\|\Gamma^2\|(1-C_6\delta)}\Big(1+\sum_{n\geq 1}\mathrm{e}^{-\frac{1}{10}nk}\Big)\\ &= \mathrm{e}^{-\beta\|\Gamma^2\|(1-C_6\delta)}\big(1+D(k)\big)\,.\end{aligned}$$

Since $\beta\|\Gamma^2\| \leq \lambda k$, by choosing A small enough and β large enough, so that δ is small enough, we have

$$(1-D(k))c_-^k \mathrm{e}^k \mathrm{e}^{-\beta\|\Gamma^2\|C_9\delta} \geq (1-D(k))c_-^k \mathrm{e}^k \mathrm{e}^{-k\lambda C_9\delta} > \mathrm{e}^{\frac{2k}{3}}$$

and

$$(1+D(k))(1+A)^k \mathrm{e}^{\frac{k}{10}} \mathrm{e}^{\beta\|\Gamma^2\|C_6\delta} \leq (1+D(k))(1+A)^k \mathrm{e}^{\frac{k}{10}} \mathrm{e}^{\lambda k C_6 \delta} < \mathrm{e}^{\frac{k}{3}}\,.$$

If these inequalities are verified, then the contribution to $-[u_\Lambda(\Gamma^q)]^{(k)}_{\mu^*}$ coming from the integrations over $\partial D^g_{r_{k,n}}$ is negligible with respect to that coming from the integrations over $\partial D^d_{r_{k,n}}$. Taking into account (A.47) we get Lemma A.6.

Lemma A.6. *There exists $0 < A' \leq A_0$ so that for all β sufficiently large, the following holds. If $k \geq K(A', \eta, \beta)$ and Γ^2 is a k-large and thin contour, then*

$$-[u_\Lambda(\Gamma^2)]^{(k)}_{\mu^*} \geq \frac{1}{20}(1-D(k))\big(\beta\Delta V(\Gamma^2)\big)^k c_-^k\, \phi^*_\Lambda(\Gamma^2)\,.$$

Proposition A.4. *There exists β' so that for all $\beta > \beta'$, the following holds. There exists an increasing diverging sequence $\{k_n\}$ such that for each k_n there exists $\Lambda(L_n)$ such that for all $\Lambda \supset \Lambda(L_n)$*

$$-[f^2_\Lambda]^{(k_n)}_{\mu^*} \geq C_{14}^{k_n}\, k_n!^{\frac{d}{d-1}}\, \Delta^{k_n} \beta^{-\frac{k_n}{d-1}}\, {\chi'_2}^{-\frac{dk_n}{d-1}}\,.$$

$C_{14} > 0$ is a constant independent of β, k_n and Λ.

Proof. We compare the contribution of the small and fat contours with that of the large and thin contours for $k \geq K(A', \eta, \beta)$. The contribution of the small contours to $|[f^2_\Lambda]^{(k)}_{\mu^*}|$ is at most

$$C_{10}\, \Delta^k\, \beta^{-\frac{k}{d-1}}\, (\hat{\theta}\chi'_2)^{-\frac{kd}{d-1}}\, k!\, k^{\frac{k}{d-1}} \leq C_{10}\, \Delta^k\, \beta^{-\frac{k}{d-1}} \Big(\frac{\mathrm{e}^{\frac{1}{d}}}{\hat{\theta}\chi'_2}\Big)^{k\frac{d}{d-1}} k!^{\frac{d}{d-1}}\,.$$

The contribution of the fat contours is much smaller by our choice of η (see (A.54)). The contribution to $-[f^2_\Lambda]^{(k)}_{\mu^*}$ of each large and thin contour is nonnegative. By assumption (A.50) and the definition of the isoperimetric constant χ_2, there exists a sequence Γ^2_n, $n \geq 1$, such that

$$\lim_{n\to\infty} \|\Gamma^2_n\| \to \infty \quad \text{and} \quad V(\Gamma^2_n)^{\frac{d-1}{d}} \geq \frac{\|\Gamma^2_n\|}{(1+\varepsilon')\chi'_2}\,.$$

Since $x^{k\frac{d}{d-1}}\mathrm{e}^{-x}$ has its maximum at $x = k\frac{d}{d-1}$, we set

$$k_n := \left\lfloor \frac{d-1}{d}\beta\|\Gamma^2_n\| \right\rfloor\,.$$

For any n, Γ^2_n is a thin and k_n-large volume contour, since by (A.51)

$$\begin{aligned}\beta\,(1-2\sqrt{A'})V(\Gamma^2)R_2(V(\Gamma^2)) &\geq \beta\,(1-2\sqrt{A'})V(\Gamma^2)^{\frac{d-1}{d}}\chi'_2 \\ &\geq \frac{(1-2\sqrt{A'})}{1+\varepsilon'}\beta\|\Gamma^2_n\| \geq k_n\,.\end{aligned}$$

If $\Lambda \supset \Gamma^2_n$, then

$$-[u_\Lambda(\Gamma^2_n)]^{(k_n)}_{\mu^*} \geq \frac{1-D(k)}{20}\left[\beta\Delta c_- V(\Gamma^2_n)\right]^{k_n}\phi^*_\Lambda(\Gamma^2_n)$$
$$\geq \frac{1-D(k)}{20}\Delta^{k_n}\beta^{-\frac{k_n}{d-1}}\Big(\frac{d\,c_-^{\frac{d-1}{d}}}{(d-1)(1+\varepsilon')\chi_2'}\Big)^{\frac{dk_n}{d-1}} k_n^{\frac{k_n d}{d-1}}\phi^*_\Lambda(\Gamma^2_n)$$

and (see (A.45))

$$k_n^{\frac{k_n d}{d-1}}\phi^*_\Lambda(\Gamma^2_n) \geq k_n^{\frac{k_n d}{d-1}}\exp\big[-\big(k_n\frac{d}{d-1}+1\big)(1+C_9\delta)\big]$$
$$\sim k_n!^{\frac{d}{d-1}}\mathrm{e}^{-C_9\delta\frac{d}{d-1}k_n}\frac{\mathrm{e}^{-1-C_9\delta}}{(2\pi k_n)^{\frac{d}{2(d-1)}}}\,.$$

By the choice (A.51) of the parameters θ and ε', if δ is small enough, i.e. β large enough, then

$$\frac{\mathrm{e}^{\frac{1}{d}}}{\theta(1-2\sqrt{A'})} < \frac{d}{d-1}\,\frac{c_-^{\frac{d-1}{d}}}{1+\varepsilon'}\mathrm{e}^{-C_9\delta}\,.$$

Hence the contributions of the small and fat contours are negligible for large k_n (see (A.53) and (A.54)). Let $\Lambda(L_n)$ be a box which contains at least $|\Lambda(L_n)|/4$ translates of Γ^2_n. For any $\Lambda \supset \Lambda(L_n)$, if k_n and β are large enough, then there exists a constant $C_{14} > 0$, independent of β, k_n and $\Lambda \supset \Lambda(L_n)$, such that

$$-[f^2_\Lambda]^{(k_n)}_{\mu^*} \geq C_{14}^{k_n}\,k_n!^{\frac{d}{d-1}}\,\Delta^{k_n}\beta^{-\frac{k_n}{d-1}}\,{\chi_2'}^{-\frac{dk_n}{d-1}}\,.$$

□

A.4.2 Bounds at Infinite Volume

We show that we can interchange the thermodynamic limit and the operation of taking the derivatives, and that the Taylor series, which exists, has a radius of convergence equal to 0. These statements are a consequence of Lemmas A.7 and A.8.

Lemma A.7. *If β is sufficiently large, then for any $k \in \mathbb{N}$ there exists $M_k = M_k(\beta) < \infty$, such that for all $t \in (\mu^* - \varepsilon, \mu^*]$ and for all finite Λ,*

$$\big|[f^2_\Lambda]^{(k)}_t\big| \leq M_k\,.$$

Proof. For sufficiently large contours, $\omega(\Gamma^2)$ is analytic and $\tau_1(\beta,\theta')$-stable on a disc of radius $\theta\Delta^{-1}R_2(V(\Gamma^2))$. From Cauchy formula

$$\big|[u_\Lambda(\Gamma^2)]^{(k)}_t\big| \leq k!\,C_{15}^k|\Gamma^2|^{\frac{k}{d-1}}\,\mathrm{e}^{-\beta\kappa|\Gamma^2|}\,,$$

for some constants C_{15} and $\kappa > 0$. Therefore, for sufficiently large contours,

$$\sum_{\Gamma^2 \subset \Lambda} \left| [u_\Lambda(\Gamma^2)]_t^{(k)} \right| \le k!\, C_{15}^k \sum_{\Gamma^2 \subset \Lambda} |\Gamma^2|^{\frac{k}{d-1}} \, \mathrm{e}^{-\beta\kappa|\Gamma^2|} \equiv |\Lambda| \beta M_k' < \infty \, .$$

This implies the existence of M_k such that $\left| [f_\Lambda^2]_t^{(k)} \right| \le M_k$. □

Lemma A.8.

$$\lim_{L\to\infty} [f^2_{\Lambda(L)}]^{(k)}_{\mu^*} = \lim_{t\uparrow\mu^*} [f]^{(k)}_t \, .$$

Proof. We compute the first derivative at the origin. Let $\eta > 0$.

$$\begin{aligned} A(\eta) :&= \frac{f(\mu^*) - f(\mu^* - \eta)}{\eta} \\ &= \lim_{L\to\infty} \frac{f^2_{\Lambda(L)}(\mu^*) - f^2_{\Lambda(L)}(\mu^* - \eta)}{\eta} \\ &= \lim_{L\to\infty} \frac{[f^2_{\Lambda(L)}]^{(1)}_{\mu^*}\,\eta + \frac{1}{2!}[f^2_{\Lambda(L)}]^{(2)}_{\mu^* - x_L(\eta)}\,\eta^2}{\eta} \\ &= \lim_{L\to\infty} \left([f^2_{\Lambda(L)}]^{(1)}_{\mu^*} + \frac{1}{2!}[f^2_{\Lambda(L)}]^{(2)}_{\mu^* - x_L(\eta)}\,\eta \right) . \end{aligned}$$

By Lemma A.7, $|[f^2_{\Lambda(L)}]^{(2)}_{\mu^* - x_L(\eta)}| \le M_2$. Therefore $\{A(\eta)\}_\eta$ is a Cauchy sequence. Hence the following limits exist,

$$[f]^{(1)}_{\mu^*} = \lim_{\eta\downarrow 0} \frac{f(\mu^*) - f(\mu^* - \eta)}{\eta} = \lim_{t\uparrow\mu^*} [f]^{(1)}_t = \lim_{L\to\infty} [f^2_{\Lambda(L)}]^{(1)}_{\mu^*} \, .$$

Same proof for the derivatives of any order.

□

Appendix B

The Stationary Phase Analysis

The following theorem is a generalisation of a result due to Isakov [Isakov1]. Let $\mathcal{D}_\rho(t) := \{z \in \mathbb{C} : |z - t| < \rho\}$.

Theorem B.1. *Let $\rho > 0$, $F(z) = \exp(-cz + bf(z))$ where $1 \le b \le c$, and f is analytic in a disc $\mathcal{D}_\rho(0)$, with a uniformly bounded derivative:*

$$\sup_{z \in \mathcal{D}_\rho(0)} |f'(z)| \le A < \frac{1}{25}\,. \tag{B.1}$$

There exists $k_0 = k_0(A)$ such that the following holds: let $t \in \mathcal{D}_\rho(0)$ and define $k_+ = (\rho - |t|)(c - 2b\sqrt{A})$. For all integer $k \in [k_0, k_+]$ there exists $z_k = r_k e^{i\varphi_k} \in \mathcal{D}_\rho(0)$ and $c_k \in \mathbb{C}$ such that

$$F^{(k)}(t) = k!\frac{c_k}{(-z_k)^k} F(-z_k + t)\,. \tag{B.2}$$

When $\operatorname{Im} t = 0$ and $f(z)$ takes real values for real z, then $\varphi_k = 0$ and $\operatorname{Im} c_k = 0$, and we have the estimates

$$\frac{3}{10}\frac{1}{\sqrt{2\pi c r_k}} < \operatorname{Re} c_k < \frac{1}{\sqrt{c r_k}} \quad \text{and} \quad \big|\operatorname{Im} c_k\big| \le \frac{1}{\sqrt{c r_k}}\,, \tag{B.3}$$

$$\big|\tan\varphi_k\big| \le \frac{bA}{c - bA} \quad \text{and} \quad \frac{k\cos\varphi_k}{c + bA} \le r_k \le \frac{k\cos\varphi_k}{c - bA}\,. \tag{B.4}$$

We have not indicated, for notational convenience, the dependence of r_k, φ_k, c_k on t. A consequence of this Theorem, for $t \in (-\rho, +\rho)$, is given after the proof in Corollary B.1. Our theorem improves significantly the original result of [Isakov1], since we show that derivatives of the function can be estimated anywhere in the disc of analyticity $\mathcal{D}_\rho(0)$. In the course of the proof, we make clear the fact that the stationary point $z_k = r_k e^{i\varphi_k}$ is solution of a *system* of equations (see (B.9)-(B.10)), whereas Isakov considered only the point $t = 0$, and there a single equation suffices to find z_k since $\varphi_k = 0$. Since this result is at the core of the proof of non-analyticity, we have explicited every step of the proof.

Proof of Theorem B.1: We use Cauchy's Formula. Define $\kappa \in (0,1)$ by $\kappa\rho = |t|$. For all $r \in (0, \rho(1-\kappa))$ we have

$$\frac{1}{k!}F^{(k)}(t) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}_r(t)} \frac{F(z)}{(z-t)^{k+1}}\,\mathrm{d}z = \frac{1}{2\pi r^k}\int_0^{2\pi} \frac{F(re^{i\varphi}+t)}{e^{ik\varphi}}\mathrm{d}\varphi\,,$$

where we have used the parametrisation $z := t + re^{i\varphi}$ for $\partial\mathcal{D}_r(t)$. Our aim is to extract the main contribution to this last integral. The integrand, because of the form of F, has a maximal value for φ close to π. We thus make a change of variable, $\varphi' := \varphi - \pi$, to obtain

$$\frac{1}{k!}F^{(k)}(t) = \frac{(-1)^k e^{-ct}}{r^k}\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{\phi(r,\varphi)+i\psi(r,\varphi)}\mathrm{d}\varphi\,, \tag{B.5}$$

where

$$\begin{aligned}\phi(r,\varphi) &:= cr\cos\varphi + b\mathrm{Re}f(-re^{i\varphi}+t)\\ \psi(r,\varphi) &:= cr\sin\varphi + b\mathrm{Im}f(-re^{i\varphi}+t) - k\varphi\,.\end{aligned}$$

If $t \in \mathbb{R}$ and f is real at real points, $\overline{f(z)} = f(\overline{z})$ and therefore $\mathrm{Im}f(\overline{z}+t) = -\mathrm{Im}f(z+t)$. By symmetry we get $F^{(k)}(t) \in \mathbb{R}$. The core of the proof is to choose r in a specific manner. This is a standard stationary phase analysis. To this end, we will need an estimate on the second derivative of f. Using Cauchy's Formula allows to obtain, for all $z_0 \in \mathcal{D}_\rho(0)$:

$$f''(z_0) = \frac{1}{2\pi i}\int_{\partial\mathcal{D}_{r'}(z_0)} \frac{f'(\omega)}{(\omega-z_0)^2}\mathrm{d}\omega = \frac{1}{2\pi}\int_0^{2\pi}\frac{f'(z_0+r'e^{i\theta})}{r'e^{i\theta}}\mathrm{d}\theta\,, \tag{B.6}$$

where $r' > 0$ is such that $|z_0| + r' < \rho$. Using the uniform bound $|f'| < A$ we obtain (we optimise taking the largest possible r', namely $\rho - |z_0|$)

$$|f''(z_0)| \le \frac{A}{\rho - |z_0|}\,. \tag{B.7}$$

Now, set $t \in \mathcal{D}_\rho(0)$, $|t| = \kappa\rho$, and consider the map $\varphi \mapsto f(-re^{i\varphi}+t)$. A direct computation yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\varphi}f(-re^{i\varphi}+t) &= -ire^{i\varphi}f'(-re^{i\varphi}+t)\\ \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}f(-re^{i\varphi}+t) &= re^{i\varphi}f'(-re^{i\varphi}+t) - (re^{i\varphi})^2 f''(-re^{i\varphi}+t)\,.\end{aligned}$$

Using (B.7) gives the bound

$$\sup_\varphi\left|\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}f(-re^{i\varphi}+t)\right| \le rA + r^2\frac{A}{\rho(1-\kappa)-r} = \frac{rA}{1-\frac{r}{\rho(1-\kappa)}}\,. \tag{B.8}$$

We now turn to the existence of a saddle point.

Lemma B.1. *Let $t \in \mathcal{D}_\rho(0)$, $|t| = \kappa\rho$. Then for all $k \in [0, \rho(1-\kappa)(c - 2b\sqrt{A})]$, the system*

$$\frac{\partial}{\partial\varphi}\phi(r,\varphi) = 0 \tag{B.9}$$

$$\frac{\partial}{\partial\varphi}\psi(r,\varphi) = 0 \tag{B.10}$$

has a solution (r_k, φ_k) with r_k and φ_k satisfying the following estimates:

$$\big|\tan\varphi_k\big| \le \frac{bA}{c - bA} \quad \textit{and} \quad \frac{k\cos\varphi_k}{c + bA} \le r_k \le \frac{k\cos\varphi_k}{c - bA}\,. \tag{B.11}$$

Proof. We make (B.9) and (B.10) explicit:

$$\sin\varphi\big(c - b\mathrm{Re}f'(-z+t)\big) - \cos\varphi\, b\mathrm{Im}f'(-z+t) = 0\,;$$
$$r\cos\varphi\big(c - b\mathrm{Re}f'(-z+t)\big) + r\sin\varphi\, b\mathrm{Im}f'(-z+t) = k\,.$$

These two equations are equivalent to

$$k\sin\varphi = rb\mathrm{Im}f'(-z+t))\,; \tag{B.12}$$

$$k\cos\varphi = r(c - b\mathrm{Re}f'(-z+t))\,. \tag{B.13}$$

Then, we see that any solution of the system (B.9), (B.10), satisfies (B.11). To show that there exists a solution, we first solve (B.13) locally for some fixed $\varphi \in (-\frac{\pi}{2}, +\frac{\pi}{2})$ (so that $\cos\varphi > 0$). Define the map $r \mapsto \xi(r,\varphi) := r(c - b\mathrm{Re}f'(-re^{i\varphi} + t))$. Since f is analytic, its real and imaginary parts are C^∞ with respect to $r > 0$ and φ (see [Rem1]), so ξ is C^∞. We have $\xi(0,\varphi) = 0$, and

$$\begin{aligned}\xi(\rho(1-\kappa),\varphi) &= \rho(1-\kappa)\big(c - b\mathrm{Re}f'\big(-\rho(1-\kappa)e^{i\varphi} + t\big)\big)\\ &\ge \rho(1-\kappa)(c - bA)\\ &\ge \rho(1-\kappa)(c - 2b\sqrt{A}) \ge k \ge k\cos\varphi\end{aligned}$$

which proves the existence of some $r_\varphi \in (0, \rho(1-\kappa)]$ such that $\xi(r_\varphi,\varphi) = k\cos\varphi$. Notice that we also have that [1]

$$\frac{r_\varphi}{\rho(1-\kappa)} = \frac{k\cos\varphi}{\rho(1-\kappa)(c - b\mathrm{Re}f')} \le \frac{c - 2b\sqrt{A}}{c - bA} < 1\,. \tag{B.14}$$

We can then show that the solution r_φ is unique, by verifying that $\frac{\partial}{\partial r}\xi(r,\varphi)$ is strictly positive at r_φ. First,

$$\begin{aligned}\frac{\partial}{\partial r}\xi(r,\varphi) &= c - b\mathrm{Re}f'(-re^{i\varphi}+t) - rb\frac{\partial}{\partial r}\mathrm{Re}f'(-re^{i\varphi}+t)\\ &= c - b\mathrm{Re}f'(-re^{i\varphi}+t) + rb\mathrm{Re}\big(\mathrm{e}^{i\varphi}f''(-re^{i\varphi}+t)\big)\\ \text{(see (B.7))} \quad &\ge c - \frac{bA}{1 - \frac{r}{\rho(1-\kappa)}}\,.\end{aligned}$$

[1]The definition of k_+, with $2\sqrt{A}$ instead of $\sqrt{A}$, ensures the strict inequality < 1.

At $r = r_\varphi$ we get (see (B.14))

$$\frac{\partial}{\partial r}\xi(r,\varphi)\Big|_{r=r_\varphi} \geq c - \frac{\sqrt{A}(c-bA)}{2-\sqrt{A}} \geq c - \frac{c\sqrt{A}}{2-\sqrt{A}} \geq \frac{8c}{9} > 0\,,$$

which proves uniqueness of r_φ. The continuity of $\varphi \mapsto r_\varphi$ is a consequence of the implicit function theorem. We turn to the second equation, and set $r = r_\varphi$. Using again equations (B.12), (B.13), we have

$$\tan\varphi = \frac{b\mathrm{Im}f'\big(r_\varphi \mathrm{e}^{i\varphi}\big)}{c - b\mathrm{Re}f'\big(r_\varphi \mathrm{e}^{i\varphi}\big)}\,.$$

On $(-\frac{\pi}{2},\frac{\pi}{2})$ the function

$$\varphi \mapsto \frac{b\mathrm{Im}f'\big(r_\varphi \mathrm{e}^{i\varphi}\big)}{c - b\mathrm{Re}f'\big(r_\varphi \mathrm{e}^{i\varphi}\big)}\,,$$

is continuous and takes its values in the interval $\big(\frac{-bA}{c-bA},\frac{bA}{c-bA}\big)$. Therefore there exists a solution [2] (r_k,φ_k), $r_k := r_{\varphi_k}$, of (B.12) and (B.13). □

Notice that we have explicit bounds on φ_k, such as

$$|\sin\varphi_k| \leq |\tan\varphi_k| \leq \frac{1}{24}\ \ ,\ \ |\cos\varphi_k| \geq \frac{4}{5}\ \ ,\ \ |\varphi_k| \leq \frac{\pi}{8}\,, \tag{B.15}$$

and that we can estimate, at $r = r_k$ (see (B.8) and (B.14)),

$$\sup_{\varphi}\left|\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}f(-r_k e^{i\varphi}+t)\right| \leq \frac{r_k A}{1-\frac{r_k}{\rho(1-\kappa)}} \leq r_k A\frac{c-bA}{2b\sqrt{A}-bA} \leq \frac{5}{9}\frac{c}{b}r_k\sqrt{A}\,. \tag{B.16}$$

We now examine (B.5) when $r = r_k$. Defining $z_k = r_k e^{i\varphi_k}$, we extract the value taken by the integrand at z_k:

$$\frac{1}{k!}F^{(k)}(t) = \frac{F(-z_k+t)}{(-z_k)^k}\underbrace{\frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{\phi(r_k,\varphi)-\phi(r_k,\varphi_k)+i\big(\psi(r_k,\varphi)-\psi(r_k,\varphi_k)\big)}\mathrm{d}\varphi}_{\equiv c_k} \tag{B.17}$$

We will estimate the integral in c_k by decomposing $[-\pi,+\pi]$ in two parts. The first is $[-\pi,+\pi]\backslash[-\frac{\pi}{4},+\frac{\pi}{4}]$.

Lemma B.2. *For all $\delta > 0$, there exists $k_1 = k_1(\delta)$ such that for all $k \geq k_1$ we have*

$$\left|\frac{1}{2\pi}\left(\int_{\frac{\pi}{4}}^{\pi}+\int_{-\pi}^{-\frac{\pi}{4}}\right)e^{\phi(r_k,\varphi)-\phi(r_k,\varphi_k)+i\big(\psi(r_k,\varphi)-\psi(r_k,\varphi_k)\big)}\mathrm{d}\varphi\right| \leq \delta\frac{1}{\sqrt{2\pi c r_k}}\,. \tag{B.18}$$

[2] We have not shown that this solution is unique.

Proof. We have $|e^{i(\psi(r_k,\varphi)-\psi(r_k,\varphi_k))}| = 1$. First, consider the interval $[\frac{\pi}{4},\pi]$. On this interval, $\cos\varphi \le y(\varphi)$ where $\varphi \mapsto y(\varphi) := \cos\frac{\pi}{4} - \sin\frac{\pi}{4}(\varphi - \frac{\pi}{4})$ (we have $y(\pi) = -0,95\dots > -1$). We can thus compute

$$\begin{aligned}
cr_k(\cos\varphi - \cos\varphi_k) &= cr_k(\cos\varphi - \cos\frac{\pi}{4} + \cos\frac{\pi}{4} - \cos\varphi_k)\\
&\le -\frac{\sqrt{2}}{2}cr_k(\varphi - \frac{\pi}{4}) + cr_k(\cos\frac{\pi}{4} - \cos\varphi_k)\\
&\le -\frac{\sqrt{2}}{2}cr_k(\varphi - \frac{\pi}{4}) - \frac{2.3}{25}cr_k\,,
\end{aligned}$$

where we used (B.15) in the last step. For the other part containing f,

$$\begin{aligned}
b\big(\operatorname{Re} f(-re^{i\varphi}+t) - \operatorname{Re} f(-re^{i\varphi_k}+t)\big) &\le b(\varphi-\varphi_k)\sup_{\varphi}\left|\frac{\mathrm{d}}{\mathrm{d}\varphi}\operatorname{Re} f(-r_k e^{i\varphi_k}+t)\right|\\
&\le br_kA(\varphi-\varphi_k)\\
&= br_kA(\varphi - \frac{\pi}{4} + \frac{\pi}{4} - \varphi_k)\\
&\le cr_kA(\varphi - \frac{\pi}{4}) + \frac{2}{25}cr_k
\end{aligned}$$

The first part of the integral can thus be bounded by:

$$\begin{aligned}
\frac{e^{-\frac{0.3}{25}cr_k}}{2\pi}\int_{\frac{\pi}{4}}^{\pi} e^{-\frac{\sqrt{2}}{2}cr_k(\varphi-\frac{\pi}{4})+cr_kA(\varphi-\frac{\pi}{4})}\mathrm{d}\varphi &\le \frac{e^{-\frac{0.3}{25}cr_k}}{2\pi}\int_0^{\infty} e^{-(\frac{\sqrt{2}}{2}-A)x}\mathrm{d}x\\
&\le \frac{e^{-\frac{0.3}{25}cr_k}}{\sqrt{2\pi cr_k}(\frac{\sqrt{2}}{2}-\frac{1}{25})}\frac{1}{\sqrt{2\pi cr_k}}\\
&\le \frac{\delta}{2}\frac{1}{\sqrt{2\pi cr_k}}\,,
\end{aligned}$$

once k is large enough, since $cr_k \ge \frac{1}{1+A}\frac{4}{5}k$ (see (B.11)). The same can be done on $[-\pi,-\frac{\pi}{4}]$, on which we use the function $y(\varphi) := \cos\frac{\pi}{4} + \sin\frac{\pi}{4}(\varphi - \frac{\pi}{4})$. □

On the interval $[-\frac{\pi}{4},+\frac{\pi}{4}]$, we use Taylor expansions for ϕ and ψ, around $\varphi = \varphi_k$. We have ($r = r_k$ is fixed)

$$\begin{aligned}
\phi(\varphi) &= \phi(\varphi_k) + 0 + \frac{1}{2!}(\varphi-\varphi_k)^2\,\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\phi\Big|_{\tilde{\varphi}}\\
\psi(\varphi) &= \psi(\varphi_k) + 0 + \frac{1}{2!}(\varphi-\varphi_k)^2\,\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\psi\Big|_{\tilde{\tilde{\varphi}}}\,,
\end{aligned}$$

where $\tilde{\varphi}$ and $\tilde{\tilde{\varphi}}$ are both functions of φ. On the interval $[-\frac{\pi}{4},+\frac{\pi}{4}]$, we have the estimates

$$-\frac{10}{9}cr_k \le \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\phi \le -\frac{5}{9}cr_k \tag{B.19}$$

Indeed, since

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\phi = -cr_k \cos\varphi + b\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\mathrm{Re}\, f(-r_k e^{i\varphi} + t)\,, \tag{B.20}$$

we use (B.16) and find

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\phi \geq -cr_k - \frac{5}{9}\sqrt{A}cr_k \geq -\frac{10}{9}cr_k\,, \tag{B.21}$$

and the upper bound

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\phi \leq -cr_k \cos\frac{\pi}{4} + \frac{5}{9}\sqrt{A}cr_k \leq -\frac{5}{9}cr_k\,. \tag{B.22}$$

We can thus compute some upper bound on the integral over $[-\frac{\pi}{4}, +\frac{\pi}{4}]$ in c_k as follows:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\frac{\pi}{4}}^{+\frac{\pi}{4}} e^{\phi(r_k,\varphi)-\phi(r_k,\varphi_k)}\mathrm{d}\varphi &\leq \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp\big(-\frac{1}{2}\cdot\frac{5}{9}cr_k x^2\big)\mathrm{d}x \\ &= \frac{3}{\sqrt{5}}\frac{1}{\sqrt{2\pi c r_k}}\end{aligned}$$

The upper bounds on $\mathrm{Re}\, c_k$ and $\mathrm{Im}\, c_k$ can be obtained by taking, say, $\delta = \frac{1}{3}$ in Lemma B.2, which gives

$$|c_k| \leq \delta\frac{1}{\sqrt{cr_k}} + \frac{3}{\sqrt{5}}\frac{1}{\sqrt{2\pi c r_k}} \leq \frac{1}{\sqrt{cr_k}}\,. \tag{B.23}$$

The lower bound on $\mathrm{Re}\, c_k$ is obtained by dividing $[-\frac{\pi}{4}, +\frac{\pi}{4}] = I_1 \cup I_2$, where $I_1 = [\varphi_k - \gamma, \varphi_k + \gamma]$, and $\gamma = \gamma_k \in [-\frac{\pi}{8}, +\frac{\pi}{8}]$ is defined by two conditions: first, fix γ small enough such that

$$\sup_{\varphi\in I_1} |\sin\varphi| \leq \frac{2A}{1-A}\,. \tag{B.24}$$

The existence of such a γ is guaranteed by (B.11). This first choice implies that

$$\begin{aligned}\sup_{\varphi\in I_1}|\psi(\varphi)-\psi(\varphi_k)| &\leq \frac{1}{2}\gamma^2 \sup_{\varphi\in I_1}\big(cr_k|\sin\varphi| + b\big|\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\mathrm{Im}\, f(-r_k e^{i\varphi}+t)\big|\big) \\ (\text{ see (B.16)}) &\leq \frac{1}{2}\gamma^2\big(cr_k\frac{2A}{1-A} + \frac{5}{9}cr_k\sqrt{A}\big) \\ &= \frac{1}{2}\gamma^2 cr_k\sqrt{A}\big(\frac{2\sqrt{A}}{1-A} + \frac{5}{9}\big) \\ &\leq \frac{1}{2}cr_k\sqrt{A}\gamma^2 \leq \frac{1}{10}cr_k\gamma^2\end{aligned}$$

Then, the second condition on γ is the following:

$$\frac{1}{10}cr_k\gamma^2 \equiv \frac{\pi}{3}\,. \tag{B.25}$$

Here, we might have to take k large enough to make sure that $\gamma \in [-\frac{\pi}{8}, +\frac{\pi}{8}]$. Then, we have a lower bound:

$$\begin{aligned}
\operatorname{Re}\frac{1}{2\pi}\int_{I_1} e^{\phi(\varphi)-\phi(\varphi_k)+i\big(\psi(\varphi)-\psi(\varphi_k)\big)}\mathrm{d}\varphi &\geq \frac{\cos\frac{\pi}{3}}{2\pi}\int_{I_1} e^{\phi(\varphi)-\phi(\varphi_k)}\mathrm{d}\varphi \\
\text{(see (B.19))} &\geq \frac{\cos\frac{\pi}{3}}{2\pi}\int_{I_1} \exp\big(-\frac{1}{2}\cdot\frac{10}{9}cr_k(\varphi-\varphi_k)^2\big)\mathrm{d}\varphi \\
&= \frac{3}{\sqrt{80\pi}}\big(2\Phi(\sqrt{\frac{100}{27}\pi})-1\big)\frac{1}{\sqrt{2\pi cr_k}} \\
&\geq \frac{47}{100}\frac{1}{\sqrt{2\pi cr_k}}
\end{aligned} \tag{B.26}$$

The upper bound on $I_2 = [-\frac{\pi}{4}, \varphi_k - \gamma] \cup [\varphi_k + \gamma, +\frac{\pi}{4}]$ is obtained easily:

$$\begin{aligned}
\frac{1}{2\pi}\Big(\int_{-\frac{\pi}{4}}^{\varphi_k-\gamma} + \int_{\varphi_k+\gamma}^{\frac{\pi}{4}}\Big)e^{\phi(\varphi)-\phi(\varphi_k)}\mathrm{d}\varphi &\leq 2\cdot\frac{1}{2\pi}\int_{\gamma}^{\frac{\pi}{4}+|\varphi_k|}\exp\big(-\frac{1}{2}\cdot\frac{5}{9}cr_kx^2\big)\mathrm{d}x \\
&\leq 2\cdot\frac{1}{2\pi}\int_{\gamma}^{+\infty}\exp\big(-\frac{1}{2}\cdot\frac{5}{9}cr_kx^2\big)\mathrm{d}x \\
&= \frac{6}{\sqrt{5}}\big(1-\Phi(\sqrt{\frac{50}{27}\pi})\big)\frac{1}{\sqrt{2\pi cr_k}} \\
&\leq \frac{3}{100}\frac{1}{\sqrt{2\pi cr_k}}
\end{aligned} \tag{B.27}$$

Taking $\delta := \frac{14}{100}$ in Lemma B.2 and using (B.26), (B.27) gives

$$\operatorname{Re}\, c_k \geq \big(\frac{47}{100} - \delta - \frac{3}{100}\big)\frac{1}{\sqrt{2\pi cr_k}} = \frac{3}{10}\frac{1}{\sqrt{2\pi cr_k}}\,, \tag{B.28}$$

which completes the proof. □

Corollary B.1. *Let $\rho > 0$, $F(z) = \exp(-cz + bf(z))$ where $1 \leq b \leq c$, and f is analytic in a disc $\mathcal{D}_\rho(0)$, taking real values on the real line, with a uniformly bounded derivative:*

$$\sup_{z\in\mathcal{D}_\rho(0)} |f'(z)| \leq A < \frac{1}{25}\,. \tag{B.29}$$

There exists $k_0 = k_0(A)$ such that the following holds: let $t \in (-\rho, +\rho)$ and define $k_+ = (\rho - |t|)(c - 2b\sqrt{A})$. For all integer $k \in [k_0, k_+]$ there exists $r_k > 0$ and $c_k > 0$ such that

$$F^{(k)}(t) = k! \frac{c_k}{(-r_k)^k} F(-r_k + t)\,. \tag{B.30}$$

We have the estimates

$$\frac{3}{10} \frac{1}{\sqrt{2\pi c r_k}} < c_k < \frac{1}{\sqrt{c r_k}}\,, \qquad \frac{k}{c + bA} \le r_k \le \frac{k}{c - bA}\,. \tag{B.31}$$

In particular, $(-1)^k F^{(k)}(t) > 0$. Moreover, if f satisfies the local condition

$$-ct + bf(t) \le -\alpha \rho c\,, \tag{B.32}$$

with $\alpha \in (\log 2, 1)$, then there exists a function $a = a(k, c, b)$, $\sup|a| < \infty$, and $\gamma = \gamma(\alpha) > 0$ such that for all $k \in [k_0, k_+]$ and A sufficiently small:

$$\big(\log(1 + F)\big)^{(k)}(t) = (1 + a \cdot e^{-\frac{\gamma}{2}k}) F^{(k)}(t) \tag{B.33}$$

Proof. The first part of the proof follows from Theorem B.1. For the second part, notice that (B.32) implies $|F(t)| \le e^{-\alpha\rho c} < 1$. The continuity of f implies that in some neighborhood $V \subset \mathcal{D}_\rho(0)$, $V \ni t$, we have $\sup_{z \in V} |F(z)| \le e^{-\frac{1}{2}\alpha\rho c} < 1$. In V we can thus use the Taylor series for $\log(1 + F)$:

$$\log\big(1 + F\big) = \sum_{n \ge 1} \frac{(-1)^{n+1}}{n} F^n = F + \sum_{n \ge 2} \frac{(-1)^{n+1}}{n} F^n$$

Since the series converges absolutely and uniformly in V, we can derive it termwise with respect to k. We then need to show that the following holds:

Lemma B.3. *Let $\alpha \in (\log 2, 1)$. There exists a positive constant $K_0 < \infty$ such that for all $\lambda \in (\frac{\log 2}{\alpha}, 1)$ and for all $n \ge 2$, $k \in [k_0, k_+]$,*

$$\left|(F^n)^{(k)}(t)\right| \le K_0 e^{-\frac{\gamma}{2}k} e^{-\alpha(1-\lambda)(n-1)k} \left|F^{(k)}(t)\right|\,, \tag{B.34}$$

where γ is given by $\gamma := \alpha\lambda - \log 2$. The constant A in (B.29) *has to be taken small enough (depending on the value of α).*

Suppose for a while that the Lemma has been shown; we have the following estimate:

$$\begin{aligned}
\left|\sum_{n \ge 2} \frac{(-1)^{n+1}}{n} (F^n)^{(k)}(t)\right| &\le \sum_{n \ge 2} \left|(F^n)^{(k)}(t)\right| \\
&\le K_0 e^{-\frac{\gamma}{2}k} \left|F^{(k)}(t)\right| \sum_{n \ge 2} e^{-\alpha(1-\lambda)(n-1)k} \\
&\le K_0 \frac{e^{-\alpha(1-\lambda)k_0}}{1 - e^{-\alpha(1-\lambda)k_0}} e^{-\frac{\gamma}{2}k} \left|F^{(k)}(t)\right|\,,
\end{aligned}$$

which proves (B.33). □

Proof of Lemma B.3: The point is that F is of exponential type. We have

$$F(z)^n = e^{-cnz+bnf(z)} \equiv e^{-c_n z+b_n f(z)}\,, \tag{B.35}$$

with $c_n = cn$, $b_n = bn$. For each $n = 1, 2, \ldots$, we can apply Theorem B.1: there exists for all $k \in [k_0, k_{n,+}]$, where $k_{n,+} = nk_+$, some $r_{n,k}$ and $c_{n,k}$ such that

$$\frac{1}{k!}(F^n)^{(k)}(t) = \frac{c_{n,k}}{(-r_{n,k})^k} F(-r_{n,k}+t)^n \tag{B.36}$$

Notice that $[k_0, k_+] \supset [k_0, k_{n,+}]$ for all n. The constant $r_{n,k}$ is a solution of the equation $k = r(c_n - b_n \mathrm{Re}\, f'(-r+t))$ and satisfies

$$\frac{k}{c_n + b_n A} \le r_{n,k} \le \frac{k}{c_n - b_n A}\,. \tag{B.37}$$

The constant $c_{n,k}$ satisfies

$$\frac{3}{10}\frac{1}{\sqrt{2\pi c_n r_{n,k}}} \le c_{n,k} \le \frac{1}{\sqrt{c_n r_{n,k}}}\,. \tag{B.38}$$

We can then consider, for all $k \in [k_0, k_+]$:

$$\frac{(F^n)^{(k)}(t)}{F^{(k)}(t)} = \frac{c_{n,k}}{c_{1,k}}\left(\frac{r_{1,k}}{r_{n,k}}\right)^k \frac{F(-r_{n,k}+t)^n}{F(-r_{1,k}+t)} \tag{B.39}$$

Notice that when n increases, $r_{n,k} \searrow 0$ and $k_{n,+} \nearrow \infty$. Using (B.37) and (B.38), we find

$$\frac{r_{1,k}}{r_{n,k}} \le n\frac{1+A}{1-A}\,, \tag{B.40}$$

and

$$\frac{c_{n,k}}{c_{1,k}} \le \frac{10}{3}\sqrt{2\pi}\sqrt{\frac{1+A}{1-A}} \equiv K_0\,. \tag{B.41}$$

We must estimate

$$\begin{aligned}\frac{F(-r_{n,k}+t)^n}{F(-r_{1,k}+t)} = \exp\big(&c(nr_{n,k}-r_{1,k}) - ct(n-1)\\ &+ b(nf(-r_{n,k}+t) - f(-r_{1,k}+t))\big)\end{aligned} \tag{B.42}$$

Using the definition of $r_{n,k}$ gives

$$\begin{aligned} nr_{n,k} - r_{1,k} &= k\left[\frac{1}{c - b\mathrm{Re}\, f'(-r_{n,k}+t)} - \frac{1}{c - b\mathrm{Re}\, f'(-r_{1,k}+t)}\right]\\ &= k\frac{b\big(\mathrm{Re}\, f'(-r_{n,k}+t) - \mathrm{Re}\, f'(-r_{1,k}+t)\big)}{(c - b\mathrm{Re}\, f'(-r_{n,k}+t))(c - b\mathrm{Re}\, f'(-r_{1,k}+t))}\\ &\le k\frac{2bA}{(c-bA)^2} \le \frac{k}{c}\frac{2A}{(1-A)^2}\end{aligned}$$

We then compute the term involving f (we use twice $|f(x) - f(x')| \leq A|x - x'|$):

$$\begin{aligned} nf(-r_{n,k}+t) - f(-r_{1,k}+t) & \qquad\qquad (B.43) \\ &= (n-1)f(t) + n(f(-r_{n,k}+t) - f(t)) - (f(-r_{1,k}+t) - f(t)) \\ &\leq (n-1)f(t) + nAr_{n,k} + Ar_{1,k} \\ &\leq (n-1)f(t) + \frac{k}{c}\frac{2A}{1-A} \end{aligned}$$

We thus have

$$\frac{F(-r_{n,k}+t)^n}{F(-r_{1,k}+t)} \leq e^{\epsilon(A)k} e^{(-ct+bf(t))(n-1)}\,, \tag{B.44}$$

where $\epsilon(A) = 2A(2-A)(1-A)^{-2}$. Since $\rho c > k_+ \geq k$, we can use assumption (B.32) and get, for all $\lambda \in (0,1)$,

$$e^{(-ct+bf(t))(n-1)} \leq e^{-\alpha k(n-1)} = e^{-\alpha\lambda k(n-1)} e^{-\alpha(1-\lambda)k(n-1)} \tag{B.45}$$

Since $\log n - \log 2 \leq \frac{1}{2}(n-2)$ for all $n \geq 1$ we can compute the following bound

$$\sup_{n\geq 2} n^k e^{-\alpha k\lambda(n-1)} = \sup_{n\geq 2} e^{k(\log n - \alpha\lambda(n-1))} \tag{B.46}$$

$$\leq \sup_{n\geq 2} e^{k(\log 2 - 1 + \alpha\lambda + n(\frac{1}{2} - \alpha\lambda))} \tag{B.47}$$

$$\leq e^{k(\log 2 - \alpha\lambda)} \equiv e^{-\zeta k}\,, \tag{B.48}$$

where we used the fact that λ is chosen such that $\zeta = \zeta(\alpha) > 0$. Putting our bounds together we bound (B.39), when A is small, by

$$K_0 \left(\frac{1+A}{1-A} e^{\epsilon(A)}\right)^k e^{-\frac{1}{2}\zeta k} e^{-\frac{1}{2}\zeta k} e^{-\alpha(1-\lambda)k(n-1)} \leq K_0 e^{-\frac{1}{2}\zeta k} e^{-\alpha(1-\lambda)k(n-1)}\,.$$

□

Appendix C

Elements of Cluster Expansion

Consider a countable set $\mathcal{D}$ whose elements are called animals, and denoted $\gamma \in \mathcal{D}$. To each animal γ is associated a finite subset of $\mathbf{Z}^d$, called the support of γ. Usually we also denote the support by γ. In the cases we consider, the support is always an R-connected set. Assume we are given a symmetric binary relation on $\mathcal{D}$, denoted $\sim$. We say two animals γ, γ' are compatible if $\gamma \sim \gamma'$. When γ and γ' are not compatible we write $\gamma \not\sim \gamma'$. We assume that the following condition is necessary to characterise incompatibility: for each animal γ, there exists a set $b(\gamma) \subset \mathbf{Z}^d$ such that if $\gamma \not\sim \gamma'$, then $b(\gamma) \cap b(\gamma') \neq \emptyset$.
To each animal $\gamma \in \mathcal{D}$ we associate a complex weight $\omega(\gamma) \in \mathbb{C}$. The partition function is defined by

$$\Xi(\mathcal{D}) := \sum_{\substack{\{\gamma\} \subset \mathcal{D} \\ \text{compat.}}} \prod_{\gamma \in \{\gamma\}} \omega(\gamma) \,, \tag{C.1}$$

where the sum extends over all sub-families of $\mathcal{D}$ of pairwise compatible animals (we assume this sum exists, which is the case in every concrete situation). When $\{\gamma\} = \emptyset$, we define the product over γ as equal to 1. We are interested in studying the logarithm of the partition function. To this end, we define the family $\hat{\mathcal{D}}$ of all maps $\hat{\gamma} : \mathcal{D} \to \{0, 1, 2, \dots\}$. The support of $\hat{\gamma}$ is the set $\{\gamma \in \mathcal{D} : \hat{\gamma}(\gamma) \geq 1\}$. Usually we also denote the support of $\hat{\gamma}$ by $\hat{\gamma}$. We will also write $\hat{\gamma} \ni x$ if the support of $\hat{\gamma}$ contains an animal whose support contains x. A map $\hat{\gamma} \in \hat{\mathcal{D}}$ is a cluster of animals if its support can't be decomposed into a disjoint union $S_1 \cup S_2$ such that each $\gamma_1 \in S_1$ is compatible with each $\gamma_2 \in S_2$. Formally, the logarithm of the partition function has the form (see e.g [Pf])

$$\log \Xi(\mathcal{D}) = \sum_{\hat{\gamma} \in \hat{\mathcal{D}}} \omega(\hat{\gamma}) \,, \tag{C.2}$$

where the weight of $\hat{\gamma}$ equals

$$\omega(\hat{\gamma}) = a^T(\hat{\gamma}) \prod_{\gamma \in \mathcal{D}} \omega(\gamma)^{\hat{\gamma}(\gamma)} \tag{C.3}$$

The numbers $a^T(\hat{\gamma})$ are purely combinatorial factors. They equal zero if $\hat{\gamma}$ is not a cluster. The following is the technical lemma that gives explicit conditions for the convergence of the development (C.2). The proof is standard and can be adapted from [Pf].

Lemma C.1. *Let* $\omega_0(\gamma)$ *be a positive weight such that*

$$\sup_{x\in\mathbf{Z}^d} \sum_{\gamma: b(\gamma)\ni x} \omega_0(\gamma) e^{|b(\gamma)|} \leq \epsilon\,, \tag{C.4}$$

where $0 < \epsilon < 1$. *Define* $\omega_0(\hat{\gamma})$ *as in* (C.3) *with* $\omega_0(\gamma)$ *in place of* $\omega(\gamma)$. *Then there exists a function* $\eta(\epsilon)$, $\lim_{\epsilon\to 0}\eta(\epsilon) = 0$ *such that*

$$\sup_{x\in\mathbf{Z}^d} \sum_{\hat{\gamma}\ni x} |\omega_0(\hat{\gamma})| \leq \eta(\epsilon)\,. \tag{C.5}$$

Typically, in the cases we consider, the weights are maps $z \mapsto \omega(\gamma; z)$, analytic in a domain $A \subset \mathbb{C}$, and there exists a positive weight $\omega_0(\gamma)$ such that $\|\omega(\gamma;\cdot)\|_A \leq \omega_0(\gamma)$ for all γ. Lemma C.1 thus implies that the series (C.2) is normally convergent, i.e. compactly convergent on A. This guarantees analyticity of the logarithm of $\Xi(\mathcal{D})$, by a standard Theorem of Weierstrass (see [Rem1], p. 249-250).

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